Controlled Fractional Dynamics and Interacting Particle Systems

Ma. Elena Hernández-Hernández
Supervisor: Prof. Vassili Kolokoltsov

Department of Statistics, University of Warwick

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Outline

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Non-Markovian Interacting Particle Systems

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- There is a final cost depending on the final state of the system
- A controller is able to influence the dynamic.

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- - - - - - **Stochastic control problem.**- - - - - - -

Aim: Minimize the final cost.

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2. **Stable laws and their domain of attraction.** A r.v. Y is a *stable* r.v. if there exist $\{a_n\}$, $a_n > 0$, and $\{b_n\}$ such that

   $$\frac{X_1 + X_2 + \cdots + X_n}{a_n} + b_n \xrightarrow{d} Y, \qquad \text{as } n \to \infty$$

   where X_n, $n \geq 0$ are i.i.d. r.v. X_1 is said to belong to the *DOA* of Y.

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   - $E|Y| = \infty$ for $\alpha \in (0,1)$.

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- **Large number of particles.** Parameters α_{ij} depend on the proportion of each type of particle:

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Result 1 (Heavy-tail semi-Markov IPS).

The system is described as a semi-Markov process with waiting times attracted by stable laws whose parameters depend on the current state.

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Semi-Markov IPS

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a) the waiting times, $\tau(N)$, are such that

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b) its kernel $F(t) = \{F_{x,y}(t)\}_{x,y\in\Sigma_d^{|N|}}$ is given by

$$F_{N,y}(t) = \begin{cases} \int_0^t \prod_{(k,l)\neq(i,j)} P\left[s \leq \tau_{kl}(n_k/|N|, n_l/|N|)\right] \nu_{ij}(ds|n_i/|N|, n_j/|N|), & y = N^{ij}, \\ 0 & else \end{cases} \tag{1}$$

where $\nu_{ij}(ds|n_i/|N|, n_j/|N|)$ is the law of $\tau_{ij}(n_i/|N|, n_j/|N|)$.

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3) *Mayer type problem*. *Only a terminal cost* given by $H : \Sigma_d^{|N|} \to \mathbb{R}^+$.

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Expected total cost. For initial state $x_0 \in \Sigma_d^{|N|}$ at time $T - t_0$, and for a control path $\tilde{u} \in \mathcal{U}$, *the cost functional* is defined by

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Objective: Minimize J by an appropriate selection of actions:

$$\inf_{\tilde{u} \in \mathcal{U}} J(t_0, x_0; \tilde{u}).$$

Dynamic Programming Approach

Given the cost functional, instead of minimizing $J(t_0, x_0, \tilde{u})$, consider the **family** of optimization problems

$$V(t,x) := \inf_{\tilde{u} \in \mathcal{U}} J(t,x,\tilde{u})$$

for $(t,x) \in [0,t_0] \times \Sigma_d^{|N|}$.

$V(\cdot,\cdot)$ is called the *value function* associated with $Y^{\tilde{u}}$.

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- - - - - - **Goal**- - - - - - -

Derive *dynamic* relationships among these problems, and *solve all of them*.

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Result 2. (Fractional HJB type equation)

The evolution equation for the payoff function of a scaled process has been *heuristically* derived and it satisfies a Fractional Partial Differential Equation.

Fractional dynamic programming equation

Let $\hat{V}(t,x)$ be the value function associated with the limiting process of a scaled version of $Y^{\tilde{u}}$. Then, under appropriate assumptions, $\hat{V}$ satisfies

$$A^*_\alpha \hat{V}(t,x) = -\hat{V}(0,x)\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{i,j=1;j\neq i}^{d} P[M^{x,u}=(i,j)]\left[\frac{\partial \hat{V}}{\partial x_j}(t,x) - \frac{\partial \hat{V}}{\partial x_i}(t,x)\right], \tag{2}$$

with initial data $V(0,x)=H(x)$, where $\alpha=\alpha(x,t,u)$, $x\in\Sigma_d$, and

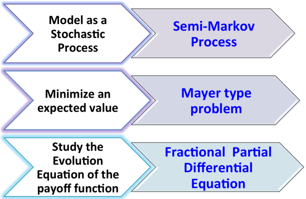
$$P[M^{x,u}=(i,j)] = \int_0^\infty \prod_{(k,l)\neq(i,j)} P[s\leq \tau_{kl}(x_k/|N|, x_l/|N|, t, u)]\nu_{ij}(ds|x_i/|N|, x_j/|N|, t, u), \tag{3}$$

and A^*_α is the dual of the generator of an $\alpha-$stable subordinator:

$$A^*_\alpha f(x) = -\frac{1}{\Gamma(-\alpha)}\int_0^\infty (f(x-s)-f(x))\frac{ds}{s^{1+\alpha}}, \quad \alpha\in(0,1).$$

Summary

The main steps for answering our questions and their implications.



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Ongoing research: Proof of the weak convergence of the underlying stochastic processes.

Thank you for attention!

References

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the Cauchy integral formula

$$I_0^n f(x) = \frac{1}{(n-1)!} \int_0^x (x-s)^{n-1} f(s) ds, \quad n > 0. \tag{5}$$

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