Homogenization of Random Media

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Understand stochastic processes in an “irregular” medium.
Asssume that the transition probabilities are still regular in a statistical sense.

Outline

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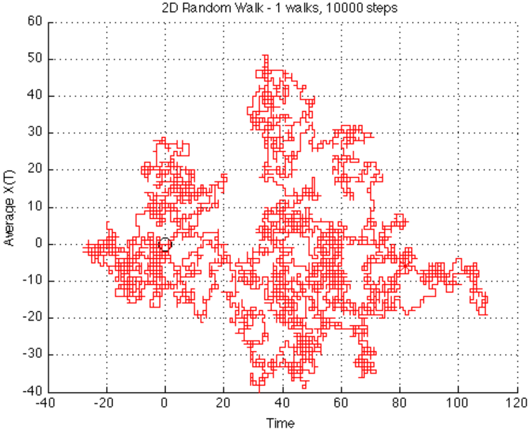
Understand stochastic processes in an “irregular” medium.
Asssume that the transition probabilities are still regular in a statistical sense.

1. Warm-up: simple random walk
2. The Random Conductance Model
3. Random walks on supercritical percolation clusters
4. Einstein relation for the Random Conductance Model
5. Einstein relation for symm. diffusions in random environment
6. Why it should be true: heuristics
7. Why it is true: strategy of the proof

Take a simple random walk on the d-dimensional lattice.



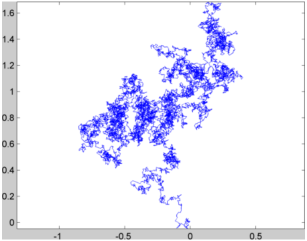
It starts from the origin and moves, with equal probabilities, to the nearest neighbours.

Simple random walk is recurrent (it returns infinitely often to the origin) if $d \in \{1, 2\}$ and it is transient (it returns only finitely often to the origin) if $d \geq 3$.

It is well-known that the scaling limit of simple random walk is a Brownian motion, a Gaussian process in continuous time on $\mathbb{R}^d$. More precisely, the law of (the linear interpolation of)

$$(X_m/\sqrt{n})_{m=0,1,\ldots,n}$$

converges to the law of $(\sigma B_t)_{0\leq t\leq 1}$ where σ is a constant depending on the dimension d. This convergence is "universal" and holds as well, for instance, for triangular lattices.



We define a random medium by giving i.i.d. weights - often called "conductances" - to the bonds of the lattice.
The, run a random walk in this medium: The transition probabilities for a point to its neighbours are proportional to the weights of the bonds.
Assume (first) that the weights are bounded and bounded away from zero.

Question

Is this walk still recurrent for $d = 2$ and transient for $d \geq 3$?

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Answer: Yes! (This was proved in in several papers by Martin Barlow, Daniel Boivin, Luis Renato Fontes/Pierre Mathieu, S. M. Kozlov, Vladas Sidoravicius/Alain-Sol Sznitman,... and extended to the case of bounded, strictly positive conductances).

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Question

How does σ depend on the law of the conductances?

Note that this is important from the viewpoint of "material sciences"! Analytical counterpart of this question, many papers but still open questions. Recent results by Antoine Gloria, Jean-Christophe Mourrat, Stefan Neukamm, Felix Otto.

Question

Can the random medium be replaced by an "averaged" deterministic medium?

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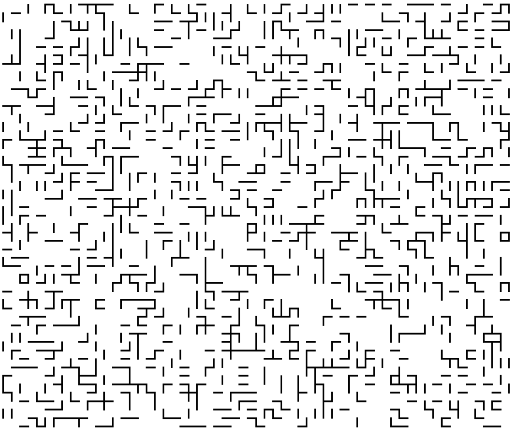
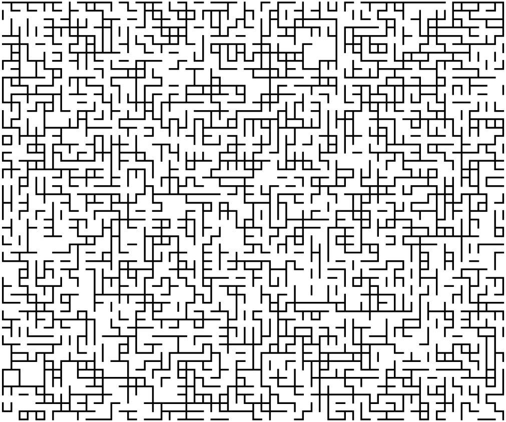
Answer: It depends!
Two (contradicting!) paradigms:
"Intermittency" versus "Homogenisation".

To be more radical, consider bond percolation with parameter p on the d-dimensional lattice: all bonds are open with probability p and closed with probability $1-p$, independently of each other. This corresponds to conductances with values either 1 or 0. Look at pictures for $d=2$:

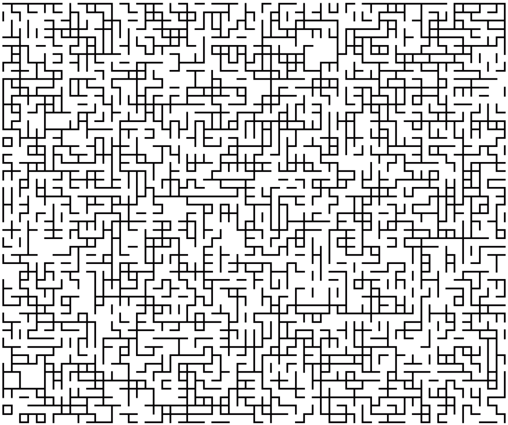
Bond percolation p= 0.25



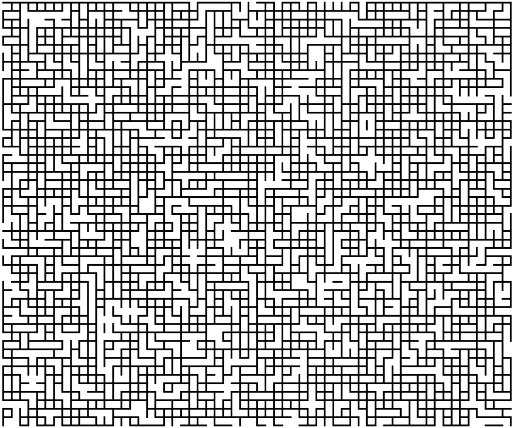
Bond percolation p= 0.49



Bond percolation p= 0.51



Bond percolation p= 0.75



For each $p \in [0,1]$, have a probability measure P_p on $\{0,1\}^{\mathbb{Z}^d}$. Let $C(0)$ be the connected open component containing the origin. Let $\theta(p) := P_p[C(0) \text{ is infinite}]$.

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Theorem

For each $d \geq 2$, there is a critical value $p_c = p_c(d) \in (0, 1)$ such that $\theta(p) = 0$ if $p < p_c$ and $\theta(p) > 0$ if $p > p_c$.

Conjecture

$\theta(p_c) = 0.$

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The conjecture has been proved for $d = 2$ and for $d \geq 19$.
(For the case $d = 2$, see the book "Probability on Graphs" by Geoffrey Grimmett.)

Take bond percolation on $\mathbb{Z}^2$, $d \geq 2$. Choose $p > p_c$ - this is called the "supercritical régime". Then, there is a unique infinite open cluster. Condition on the event that the origin is in the infinite cluster.
Start a random walk in the infinite cluster which can only walk on open bonds, and which goes with equal probabilities to all neighbours. (In particular, this random walk never leaves the infinite cluster.)

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Answer: Yes! (The recurrence part follows again from Rayleigh's Monotonicity Principle, the transience was shown by Geoffrey Grimmett, Harry Kesten and Yu Zhang).

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Answer: Yes! (This was proved by Noam Berger/Marek Biskup, Pierre Mathieu/Andrey Piatnitski, Vladas Sidoravicius/Alain-Sol Sznitman). Method of proof: decompose the walk in a martingale part and a "corrector". Show that the corrector can be neglected and apply the CLT for martingales. The corrector is an interesting process, see Jean-Christophe Mourrat/Felix Otto for recent results.

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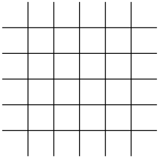
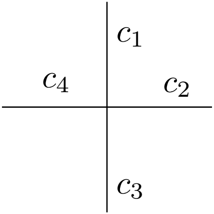
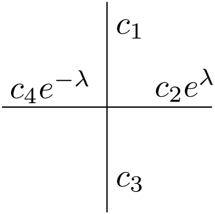
Consider again the Random Conductance Model:
Take i.i.d. conductances on the bonds of the d-dimensional lattice, bounded above and bounded away from 0.



Known that there is a Central Limit Theorem (even an invariance principle), many recent papers (Martin Barlow, Marek Biskup/Tim Prescott, Luis RenatoFontes/Pierre Mathieu, Noam Berger/Marek Biskup/Christopher Hoffman/Gady Kozma, Pierre Mathieu, Vladas Sidoravicius/Alain-Sol Sznitman, ...).

We denote the covariance matrix by Σ.
Add a drift: take $\lambda > 0$ and multiply the conductances with powers of e^{λ}.





Theorem

(Lian Shen 2002)
For fixed drift, there is a law of large numbers:
For any $\lambda > 0$,

$$\lim_{n\to\infty} \frac{1}{n} X^{\lambda,\omega}(n) = v(\lambda), \ \textit{a.s.}$$

where $v(\lambda)$ *is deterministic and* $v(\lambda) \cdot e_1 > 0$.

Theorem

Einstein relation (N. G., Jan Nagel and Xiaoqin Guo, in progress)

$$\lim_{\lambda \to 0} \frac{v(\lambda)}{\lambda} = \Sigma e_1 \, .$$

The theorem has been proved by Tomasz Komorowski and Stefano Olla (2005) in the case where $d \geq 3$ and the conductances only take two values.

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The theorem has been proved by Tomasz Komorowski and Stefano Olla (2005) in the case where $d \geq 3$ and the conductances only take two values. Of course, it can be verified in the one-dimensional case and in the deterministic case.

Consider diffusion $X^\omega(t)$ in $\mathbb{R}^d$ with generator

$$L^\omega f(x) = \frac{1}{2} e^{2V^\omega(x)} \operatorname{div}(e^{-2V^\omega} a^\omega \nabla f)(x), \tag{1}$$

where V^ω is a real function and a^ω is symmetric matrix. V^ω and a^ω are realizations of a random environment, defined on some prob. space $(\Omega, \mathcal{A}, Q)$.

Assumptions:
(1) Translation invariance, ergodicity
(2) Smoothness: $x \to V^\omega(x)$ and $x \to a^\omega(x)$ are smooth
(3) Uniform ellipticity: V^ω is bounded and a^ω is uniformly elliptic, namely there exists a constant κ such that, for all ω, $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^d$,

$$\kappa|y|^2 \leq |a^\omega(x)y|^2 \leq \kappa^{-1}|y|^2 \,.$$

(4) Finite range dependence.

Then, with $\sigma^\omega = \sqrt{a^\omega}$ and $b^\omega = \frac{1}{2}\mathrm{div} a^\omega - a^\omega \nabla V^\omega$, X^ω solves the stochastic differential equation

$$dX^\omega(t) = b^\omega(X^\omega(t))\, dt + \sigma^\omega(X^\omega(t)) dW_t \tag{2}$$

where W is a Brownian motion.

Theorem

(George Papanicolaou, Srinivasa Varadhan, Hirofumi Osada, S. M. Kozlov 1980, 1982) The process X^ω satisfies a Central Limit Theorem i.e. $\frac{1}{\sqrt{t}}X^\omega(t)$ converges in law towards a Gaussian law. More precisely, the rescaled process

$$X_\varepsilon^\omega(t) := \varepsilon X^\omega(t/\varepsilon^2)\,,\ t \geq 0 \tag{3}$$

satisfies an invariance principle: there exists a non-negative (deterministic) symmetric matrix Σ such that the law of $(X_\varepsilon^\omega(t))_{t\geq 0}$ converges to the law of $(\sqrt{\Sigma}\,W(t))_{t\geq 0}$.

Both statements hold for almost any realization of the environment. Note that Σ is in general **not** the average of a^ω.

Now, add a local drift in the equation satisfied by X^ω: let $\ell \in \mathbb{R}^d$ be a vector, $\ell \neq 0$, and take the equation

$$dX^{\lambda,\omega}(t) = b^\omega(X^{\lambda,\omega}(t))dt + \sigma^\omega(X^{\lambda,\omega}(t))dW_t + a^\omega(X^{\lambda,\omega}(t))\lambda\ell dt. \quad (4)$$

Theorem

(Lian Shen 2003)
Assume Q has finite range of dependence and V^ω is smooth and bounded. Then the diffusion in random environment $X^{\lambda,\omega}$ satisfies the law of large numbers: For any $\lambda > 0$,

$$\lim_{t\to\infty} \frac{1}{t} X^{\lambda,\omega}(t) = v(\lambda), \ a.s. \tag{5}$$

where $v(\lambda)$ is a deterministic vector and $\ell \cdot v(\lambda) > 0$.

v is called the effective drift.

Strategy of the proof (for fixed λ): Show that the process is transient in direction ℓ,

$$\lim_{t} \ell \cdot X^{\lambda,\omega}(t) = +\infty\,, \text{ a.s.} \tag{6}$$

Define regeneration times $\tau_1, \tau_2, \ldots$ Show that $\mathbb{E}_0^\lambda\left[\tau_2 - \tau_1\right] < \infty$. Conclude that

$$\lim_{t\to\infty} \frac{1}{t} X^{\lambda,\omega}(t) = \frac{\mathbb{E}_0^\lambda\left[X^{\lambda,\omega}(\tau_2) - X^{\lambda,\omega}(\tau_1)\right]}{\mathbb{E}_0^\lambda\left[\tau_2 - \tau_1\right]} \quad \text{a.s.} \tag{7}$$

Theorem

Einstein relation. (N.G., Pierre Mathieu, Andrey Piatnitski)
The effective diffusivity can be interpreted with the derivative of the effective drift:

$$\lim_{\lambda \to 0} \frac{v(\lambda)}{\lambda} = \Sigma \ell \,. \tag{8}$$

In other words, the function $\lambda \to v(\lambda)$ has a derivative at 0 and we have for any vector e

$$\lim_{\lambda \to 0} \frac{1}{\lambda} e \cdot v(\lambda) = e \cdot \Sigma \ell. \tag{9}$$

A key ingredient is **Girsanov transform**. For any t, the law of $(X^{\lambda,\omega}(s))_{0\leq s\leq t}$ is absolutely continuous w. r. t. the law of $(X^{\omega}(s))_{0\leq s\leq t}$ and the Radon-Nikodym density is the exponential martingale

$$e^{\lambda B^{\omega}(t)-\frac{\lambda^2}{2}\langle B^{\omega}\rangle(t)} \tag{10}$$

where

$$B^{\omega}(t)=\int_0^t \ell^T\sigma^{\omega}(X^{\omega}(s))\cdot dW_s \tag{11}$$

and

$$\langle B^{\omega}\rangle(t)=\int_0^t \left|\ell^T\sigma^{\omega}(X^{\omega}(s))\right|^2 ds \tag{12}$$

In particular,

$$\mathbb{E}_0^{\lambda}\left[X^{\lambda,\omega}(t)\right] = \mathbb{E}_0\left[X^{\omega}(t)e^{\lambda B^{\omega}(t)-\frac{\lambda^2}{2}\langle B^{\omega}\rangle(t)}\right] \tag{13}$$

Hence

$$\frac{d}{d\lambda}\,\mathbb{E}_0^{\lambda}\left[X^{\lambda,\omega}(t)\right]\Big|_{\lambda=0} = \mathbb{E}_0\left[X^{\omega}(t)B^{\omega}(t)\right] \tag{14}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\frac{d}{d\lambda}\,\mathbb{E}_0^{\lambda}\left[X^{\lambda,\omega}(t)\right]\Big|_{\lambda=0} = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}_0\left[X^{\omega}(t)B^{\omega}(t)\right] \tag{15}$$

Exchanging the order of the limits yields

$$\frac{d}{d\lambda}\, v(\lambda)\big|_{\lambda=0} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0\left[X^\omega(t)B^\omega(t)\right] \tag{16}$$

A symmetry argument (using the reversibility) shows that

$$\mathbb{E}_0\left[X^\omega(t)B^\omega(t)\right] = \mathbb{E}_0\left[X^\omega(t)(\ell\cdot X^\omega(t))\right] \tag{17}$$

and we conclude that

$$\frac{d}{d\lambda}\, v(\lambda)\big|_{\lambda=0} = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_0\left[X^\omega(t)(\ell\cdot X^\omega(t))\right] = \Sigma\ell \tag{18}$$

In fact, Joel Lebowitz and Hermann Rost showed, using the invariance principle and Girsanov transform:

Theorem

(Joel Lebowitz, Hermann Rost, 1994)
Let $\alpha > 0$. *Then*

$$\lim_{\lambda \to 0\,, t \to +\infty\,, \lambda^2 t = \alpha} \mathbb{E}_0^{\lambda}\left[\frac{X^{\lambda,\omega}(t)}{\lambda t}\right] = \Sigma\,\ell\,. \tag{19}$$

Idea: work on the scale $\lambda \to 0$, $t \to \infty$, $\lambda^2 t \to \alpha$ and eventually $\alpha \to \infty$.
We show that

Proposition

$$\lim_{\alpha \to +\infty} \limsup_{\lambda \to 0\,, t \to +\infty\,, \lambda^2 t = \alpha} \left| \mathbb{E}_0^{\lambda} \left[\frac{X^{\lambda,\omega}(t)}{\lambda t} \right] - \frac{v(\lambda)}{\lambda} \right| = 0. \tag{20}$$

Recall that

$$\lim_{\lambda \to 0, t \to +\infty, \lambda^2 t = \alpha} \mathbb{E}_0^{\lambda}\left[\frac{X^{\lambda,\omega}(t)}{\lambda t}\right] = \Sigma\, \ell\,. \tag{21}$$

In order to show the proposition, follow Lian Shen's construction of regeneration times, but take into account the dependence on λ. To carry this through, need uniform estimates for hitting times (on our scale).

The Einstein relation is conjectured to hold for many models, but it is proved for few. Examples include:
–Balanced random walks in random environment (Xiaoqin Guo).
–Random walks on Galton-Watson trees (Gérard Ben Arous, Yueyun Hu, Stefano Olla, Ofer Zeitouni).
–Tagged particle in asymmetric exclusion (Michail Loulakis).
–Random walks on one-dimensional percolation clusters (N.G., Matthias Meiners, Sebastian Müller, in progress).

Open questions: plenty!
But instead:

Thanks for your attention!