# Stochastic calculus in Banach spaces, an infinite dimensional PDE and stability results

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# Plan of the talk

1. A first motivation: calculus for path dependent random elements and a representation result.
   - Alternative approach to Dupire and Cont, R. and Fournié, D.
2. Stochastic calculus and Itô formula for general Banach space valued stochastic processes based on generalized notion of quadratic variation.
   - Among the classical theory Da Prato, G. and Zabczyk J., Métivier M. and Pellaumail J., Dinculeanu N., Prévôt C. and Röckener M. but also Z. Brzezniak, J. Van Neerven, M. Riedle, M. Veraar, L.Weis ...

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# Robustness of Black-Scholes formula

Let $W$ be the real Brownian motion equipped with its canonical filtration $(\mathcal{F}_t)$, $\langle W \rangle_t = t$.
Let $S$ be the price of a financial asset

$$S_t = \exp(\sigma W_t - \frac{\sigma^2}{2} t)\, , \sigma > 0\, .$$

Let $\tilde{f}, f : \mathbb{R} \to \mathbb{R}$ and

$$h := \tilde{f}\,(S_T) = f(W_T)$$

where $f(y) = \tilde{f}\left(\exp(\sigma y - \frac{\sigma^2}{2} T)\right)$.

Let $\tilde{u} : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ solving

$$\begin{cases} \partial_t \tilde{u}(t,x) + \frac{1}{2}\partial_{xx}\tilde{u}(t,x) = 0 & \\ \tilde{u}(T,x) = \tilde{f}(x) & x \in \mathbb{R} \end{cases}$$

Applying classical Itô formula we obtain

$$h = \tilde{u}(0, S_0) + \int_0^T \partial_x \tilde{u}(s, S_s) dS_s$$

or even

$$h = u(0, W_0) + \int_0^T \partial_x u(s, W_s) dW_s$$

for a suitable $u : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$.

# A toy model for $X$ real valued

Does one have a similar formula if $W$ is replaced by a finite quadratic variation $X$ such that $[X]_t = t$ but not necessarily a semimartingale? The answer is YES.

Let $X$ such that $[X]_t = t$

A1 $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and polynomial growth

A2 $v \in C^{1,2}([0, T] \times \mathbb{R})$ such that

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\partial_{xx} v(t,x) = 0 \\ v(T,x) = f(x) \end{cases}$$

Then the r.v. $h := f(X_T)$ admits following representation

$$h := v(0, X_0) + \int_0^T \partial_x v(s, X_s) d^- X_s$$

Schoenmakers-Kloeden (1999) Coviello-Russo (2006)

# A generalized Clark-Ocone formula

- We suppose that the law of $X = W$ is not anymore a Wiener measure but $X$ is still a finite quadratic variation process (not necessarily a semimartingale) such that

$$[X]_t = \int_0^t \sigma^2(s, X_s)ds, \qquad \sigma : [0, T] \times \mathbb{R} \to \mathbb{R}^+$$

- Are there reasonable classes of random variable which can be represented in the form

$$h = H_0 + \text{"} \int_0^T \xi_s dX_s \text{"?}$$

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## Natural question

Is it possible to express generalization of it where the option is path dependent?

## Definition

Let $T > 0$ and $X = (X_t)_{t \in [0,T]}$ be a real continuous process prolongated by continuity.
Process $X(\cdot)$ defined by

$$X(\cdot) = \{X_t(u) := X_{t+u}; u \in [-T, 0]\}$$

will be called **window process**.

- $X(\cdot)$ is a $C([-T, 0])$-valued stochastic process.
- $C([-T, 0])$ is a typical non-reflexive Banach space.

# The toy model revisited

As first step we revisit the toy model.

## Proposition

We set $\eta \in C([-T,0])$ and we define

- $H : C([-T,0]) \longrightarrow \mathbb{R}$, by $H(\eta) := f(\eta(0))$
- $u : [0,T] \times C([-T,0]) \longrightarrow \mathbb{R}$, by $u(t,\eta) := v(t,\eta(0))$

Then

$$u \in C^{1,2}([0,T] \times C([-T,0]))$$

and solves an infinite dimensional PDE

$$\begin{cases} \partial_t u(t,\eta) + \frac{1}{2}\langle D^2 u(t,\eta)\ ,\ \mathbb{1}_D\rangle = 0 \\ u(T,\eta) = H(\eta) \end{cases}$$

**Proof.**

- $u(T, \eta) = v(T, \eta(0)) = f(\eta(0)) = H(\eta)$
- $\partial_t u(t, \eta) = \partial_t v(t, \eta(0))$
- $Du(t, \eta) = \partial_x v(t, \eta(0))\, \delta_0$
- $D^2 u(t, \eta) = \partial^2_{x\,x} v(t, \eta(0))\, \delta_0 \otimes \delta_0$ $\qquad D^2 u(t, \eta) \in \mathcal{D}_{0,0}$
- $\partial_t u(t, \eta) + \frac{1}{2} D^2 u(t, \eta)(\{0, 0\}) = 0$

$\square$

And, let $X$ such that $[X]_t = t$, we have

$$h := H(X_T(\cdot)) = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s$$

# A representation problem

We suppose $X_0 = 0$ and $[X]_t = \int_0^t \sigma^2(s, X_s)ds$.
The main task will consist in looking for classes of functionals

$$H : C([-T, 0]) \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits representation

$$h = H_0 + \int_0^T \xi_s d^- X_s$$

- Moreover we look for an explicit expression for
  - $H_0 \in \mathbb{R}$
  - $\xi$ adapted process with respect to the canonical filtration of $X$

# Idea

We will obtain the representation formula by expressing $h = H(X_T(\cdot))$ as

$$h = H(X_T(\cdot)) = \lim_{t \uparrow T} u(t, X_t(\cdot))$$

where $u \in C^{1,2}([0,T[\times C([-T,0]))$ solves an infinite dimensional PDE, if previous limit exists.

## Representation of $h = H(X_T(\cdot))$

Then

$$h = u(0, X_0(\cdot)) + \int_0^T D^{\delta_0} u(s, X_s(\cdot)) d^- X_s \tag{1}$$

where $D^{\delta_0} u(s,\eta) = D\,u(s,\eta)(\{0\})$ is the *projection* of the Fréchet derivative $Du\,(t,\eta)$ on the linear space generated by Dirac measure $\delta_0$, we recall that
$D\,u : [0,T] \times C([-T,0]) \longrightarrow C^*([-T,0]) = \mathcal{M}([-T,0])$.

# Forward integral for real valued processes

## Definition

Let $X$ (resp. $Y$) be a continuous (resp. locally integrable) process. Suppose that the random variables

$$\int_0^t Y_s d^- X_s := \lim_{\epsilon \to 0} \int_0^t Y_s \frac{X_{s+\epsilon} - X_s}{\epsilon} ds$$

exists ucp, then the limiting process denoted by $\int_0^\cdot Y d^- X$ is called the **(proper) forward integral of $Y$ with respect to $X$**.

Russo-Vallois 1993

## Remark

If $S$ semimartingale and $Y$ cadlag and predictable

$$\int_0^\cdot Y d^- S = \int_0^\cdot Y dS \quad \text{(Itô)}$$

# Covariation for real valued processes

## Definition

The **covariation of** $X$ **and** $Y$ is defined by

$$[X, Y]_t = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t (X_{s+\epsilon} - X_s)(Y_{s+\epsilon} - Y_s) ds$$

if the limit exists in the ucp sense with respect to $t$.
If $X = Y$, $X$ is said to be **finite quadratic variation process** and $[X] := [X, X]$.

## Remark

Let $S^1$, $S^2$ be $(\mathcal{F}_t)$-semimartingales with decomposition $S^i = M^i + V^i$, $i = 1, 2$

- $[S^i]$ classical bracket and $[S^i] = \langle M^i \rangle$.
- $[S^1, S^2]$ classical bracket and $[S^1, S^2] = \langle M^1, M^2 \rangle$.

# Itô formula for finite quadratic variation processes

## Theorem

Let $F : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $F \in C^{1,2}\left([0,T[\times\mathbb{R}\right)$ and $X$ be a finite quadratic variation process. Then

$$\int_0^t \partial_x F(s, X_s) d^- X_s$$

exists in the ucp sense and equals

$$F(t, X_t) - F(0, X_0) - \int_0^t \partial_s F(s, X_s) ds - \frac{1}{2} \int_0^t \partial_{x\,x} F(s, X_s) d[X]_s$$

# An infinite dimensional framework

We fix now in a general (infinite dimensional) framework. Let

- $B$ general Banach space
- $\mathbb{X}$ a $B$-valued process
- $u : [0, T] \times B \longrightarrow \mathbb{R}$ be of class $C^{1,2}$ in Fréchet sense.

## An Ito formula for $B$-valued processes

We obtain an Itô type expansion of $u(t, \mathbb{X}_t)$, available also for $B = C([-T, 0])$-valued processes, as window processes, i.e. when $\mathbb{X} = X(\cdot)$.

$u : [0, T] \times B \longrightarrow \mathbb{R}$ be of class $C^{1,2}$ in Fréchet sense, then

- $Du : [0, T] \times B \longrightarrow L(B; \mathbb{R}) := B^*$;
- $D^2u : [0, T] \times B \longrightarrow L(B; B^*) \cong \mathcal{B}(B \times B) \cong (B\hat{\otimes}_\pi B)^*$

where

- $\mathcal{B}(B \times B)$ Banach space of real valued bounded bilinear forms on $B \times B$
- $(B\hat{\otimes}_\pi B)^*$ dual of the tensor projective tensor product of $B$ with $B$.
- $B\hat{\otimes}_\pi B$ fails to be Hilbert even if $B$ is a Hilbert space (is not even a reflexive space).

# A first attempt to an Itô type expansion of $u(\cdot, \mathbb{X})$

$$
\begin{aligned}
u(t, \mathbb{X}_t) = u(0, \mathbb{X}_0) + \int_0^t \partial_s u(s, \mathbb{X}_s) ds + \\
+ '' \int_0^t {}_{B^*}\langle Du(s, \mathbb{X}_s), d\mathbb{X}_s \rangle_B '' + \\
+ \frac{1}{2}'' \int_0^t {}_{(B\hat{\otimes}_\pi B)^*}\langle D^2 u(s, \mathbb{X}_s), d[\mathbb{X}]_s \rangle_{B\hat{\otimes}_\pi B} ''
\end{aligned}
$$

The literature does not apply: several problems appear even in the simple case $W(\cdot)$!

# Stochastic calculus via regularization for Banach valued processes

Let $B$ be a Banach space and $\mathbb{X}$ a $B$-valued stochastic process.

**We will define**

- a stochastic integral for $B^*$-valued integrand with respect to $B$-valued integrators, which are not necessarily semimartingale.
- a new concept of quadratic variation which involves a Banach subspace $\chi$ continuously injected into $(B\hat{\otimes}_\pi B)^*$. It will be called $\chi$-quadratic variation of $\mathbb{X}$.

# Forward integral for Banach valued processes

## Definition

Let $\mathbb{X}$ and $\mathbb{Y}$ be respectively a $B$-valued and a $B^*$-valued continuous stochastic processes.
If the process defined for every fixed $t \in [0, T]$ by

$$\int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s \rangle_B := \lim_{\epsilon \to 0} \int_0^t {}_{B^*}\langle \mathbb{Y}(s), \frac{\mathbb{X}(s+\epsilon) - \mathbb{X}(s)}{\epsilon} \rangle_B \, ds$$

in probability admits a continuous version, then process

$$\left( \int_0^t {}_{B^*}\langle \mathbb{Y}_s, d^-\mathbb{X}_s \rangle_B \right)_{t \in [0,T]}$$

will be called **forward stochastic integral of $\mathbb{Y}$ with respect to** $\mathbb{X}$.

# Definition of Chi-subspace

## Definition

A Banach subspace $\chi$ continuously injected into $(B\hat{\otimes}_\pi B)^*$ will be called a **Chi-subspace of** $(B\hat{\otimes}_\pi B)^*$.
In particular it holds

$$\|\cdot\|_\chi \geq \|\cdot\|_{(B\hat{\otimes}_\pi B)^*}.$$

## Remark

If $\chi \subset (B\hat{\otimes}_\pi B)^*$ then $B\hat{\otimes}_\pi B \subset (B\hat{\otimes}_\pi B)^{**} \subset \chi^*$

# Definition of Chi-quadratic variation

## Definition

$\mathbb{X}$ **admits a** $\chi$**-quadratic variation** if

H1 For all $(\epsilon_n) \downarrow 0$ it exists a subsequence $(\epsilon_{n_k})$ such that

$$\sup_k \int_0^T \frac{\left\| J\left( (\mathbb{X}_{s+\epsilon_{n_k}} - \mathbb{X}_s) \otimes^2 \right) \right\|_{\chi^*}}{\epsilon_{n_k}} ds \quad < \infty \quad a.s.$$

H2 where $[\mathbb{X}]^\epsilon : \chi \longrightarrow \mathscr{C}([0,T])$ defined by

$$\phi \mapsto \left( \int_0^t {}_\chi\langle \phi, \frac{J\left( (\mathbb{X}_{s+\epsilon} - \mathbb{X}_s) \otimes^2 \right)}{\epsilon} \rangle_{\chi^*} \, ds \right)_{t \in [0,T]} .$$

There exists $[\mathbb{X}] : \chi \longrightarrow \mathscr{C}([0,T])$ such that

$$[\mathbb{X}]^\epsilon(\phi) \xrightarrow[\epsilon \to 0]{ucp} [\mathbb{X}](\phi) \quad \forall \, \phi \in \chi$$

# $\chi$-quadratic variation and global quadratic variation concept

## Definition

The $\chi^*$-valued bounded variation process $\widetilde{[\mathbb{X}]}$, defined by $\widetilde{[\mathbb{X}]}_t(\phi) = [\mathbb{X}](\phi)_t$ a.s. for all $\phi \in \chi$ is called **$\chi$-quadratic variation of $\mathbb{X}$**.

## Definition

We say that $\mathbb{X}$ admits a **global quadratic variation (g.q.v.)** if it admits a $\chi$-quadratic variation with $\chi = (B\hat{\otimes}_\pi B)^*$.

# Infinite dimensional Itô's formula

Let $B$ a separable Banach space

## Theorem (Itô's formula)

Let $\mathbb{X}$ a $B$-valued continuous process admitting a $\chi$-quadratic variation.
Let $u : [0, T] \times B \longrightarrow \mathbb{R}$ be $C^{1,2}$ Fréchet such that

$$D^2 u : [0, T] \times B \longrightarrow \chi \subset (B \hat{\otimes}_\pi B)^* \quad \text{continuously}$$

Then for every $t \in [0, T]$ the forward integral

$$\int_0^t {}_{B^*}\langle Du(s, \mathbb{X}_s), d^- \mathbb{X}_s \rangle_B$$

exists and following formula holds.

## Ito's formula

$$u(t,\mathbb{X}_t) = u(0,\mathbb{X}_0) + \int_0^t \partial_s u(s,\mathbb{X}_s) ds + \\ + \int_0^t {}_{B^*}\langle Du(s,\mathbb{X}_s), d^-\mathbb{X}_s\rangle_B + \\ + \frac{1}{2}\int_0^t {}_{\chi}\langle D^2 u(s,\mathbb{X}_s), d\widetilde{[\mathbb{X}]}_s\rangle_{\chi^*}$$

# Window processes

- We fix attention now on $B = C([-T,0])$-valued window processes.
- $X$ continuous real valued process and $X(\cdot)$ its window process.
- $\mathbb{X} = X(\cdot)$

# Evaluations of $\chi$-quadratic variation for window processes

- If $X$ has Hölder continuous paths of parameter $\gamma > 1/2$, then $X(\cdot)$ has a zero g.q.v.
  For instance:
  - $X = B^H$ fractional Brownian motion with parameter $H > 1/2$.
  - $X = B^{H,K}$ bifractional Brownian motion with parameters $H \in ]0, 1[$, $K \in ]0, 1]$ s.t. $HK > 1/2$.
- $W(\cdot)$ does not admit a g.q.v.

# Examples of Chi-subspaces and evaluations of $\chi$-quadratic variation for window processes

- $\chi$ Chi-subspace of $(B\hat{\otimes}_\pi B)^*$. For instance:
  - $\mathcal{M}([-T,0]^2)$ equipped with the total variation norm.
  - $L^2([-T,0]^2)$.
  - $\mathcal{D}_{0,0} = \{\mu(dx,dy) = \lambda\,\delta_0(dx)\otimes\delta_0(dy)\}$.
  - $\left(\mathcal{D}_0 \oplus L^2\right)\hat{\otimes}_h^2 = \mathcal{D}_{0,0} \oplus L^2([-T,0])\hat{\otimes}_h D_0 \oplus D_0\hat{\otimes}_h L^2([-T,0]) \oplus L^2([-T,0]^2)$.
  - $Diag := \{\mu(dx,dy) = g(x)\delta_y(dx)dy;\, g \in L^\infty([-T,0])\}$.
- $W(\cdot)$ does not admit a $\mathcal{M}([-T,0]^2)$-quadratic variation.
- If $X$ is a real finite quadratic variation process, then $X(\cdot)$ admits a $\chi$-quadratic variation for all previous $\chi$ explicitly given in term of the quadratic variation $[X]$.

# Evaluations of $\chi$-quadratic variation for window processes

- $X(\cdot)$ has zero $L^2([-T,0]^2)$-quadratic variation.
- $X(\cdot)$ has $\mathcal{D}_{0,0}$-quadratic variation

$$[X(\cdot)] : \mathcal{D}_{0,0} \longrightarrow \mathscr{C}[0,T] , \qquad [X(\cdot)]_t(\mu) = \mu(\{0,0\})[X]_t$$

- $X(\cdot)$ has $\left(\mathcal{D}_0 \oplus L^2\right) \hat{\otimes}_h^2$-quadratic variation

$$[X(\cdot)] : \left(\mathcal{D}_0 \oplus L^2\right) \hat{\otimes}_h^2 \longrightarrow \mathscr{C}[0,T] , \qquad [X(\cdot)]_t(\mu) = \mu(\{0,0\})[X]_t$$

- $X(\cdot)$ has *Diag*-quadratic variation

$$[X(\cdot)] : Diag \longrightarrow \mathscr{C}[0,T] , \qquad [X(\cdot)]_t(\mu) = \int_0^t g(-x)[X]_{t-x} dx$$

where $\mu(dx,dy) = g(x)\delta_y(dx)dy$.

# We come back to a representation problem

Let $B = C([-T, 0])$, $\eta \in B$. The main task will consist in looking for classes of functionals

$$H : B \longrightarrow \mathbb{R}$$

such that the r.v.

$$h := H(X_T(\cdot))$$

admits representation

$$h = H_0 + \int_0^T \xi_s d^- X_s$$

- Moreover we look for an explicit expression for
  - $H_0 \in \mathbb{R}$
  - $\xi$ adapted process with respect to the canonical filtration of $X$

# An infinite dimensional PDE

Let $H : B \longrightarrow \mathbb{R}$. We show the existence of $u : [0, T] \times B \longrightarrow \mathbb{R}$ of class $C^{1,2}([0, T[\times B) \cap C^0([0, T] \times B)$ solving the following infinite dimensional PDE

$$\begin{cases} \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^{-}\eta(x) + \frac{1}{2}\int_{D_t} D^2_{dx\,dy} u(t,\eta)\sigma^2(t,\eta(x)) = 0 \\ u(T,\eta) = H(\eta) \end{cases} \tag{2}$$

where $D_t := \{(x,x), x \in [-t,0]\}$ and

- $D^{\perp}_{dx} u(t,\eta) := Du(t,\eta) - Du(t,\eta)(\{0\})$.
- If $D^{\perp}_{dx} u(t,\eta)$ admits a density which is absolutely continuous we denote its density by $x \mapsto D^{ac}_x u(t,\eta)$.
- If $x \mapsto D^{ac}_x u(t,\eta)$ has bounded variation, previous integral coincides with the one defined by an integration by parts.

## Representation of $h = H(X_T(\cdot))$ with $[X]_t = \int_0^t \sigma^2(s, X_s)ds$

Then

$$h = H_0 + \int_0^T \xi_s d^- X_s \tag{3}$$

with

- $H_0 = u(0, X_0(\cdot))$
- $\xi_s = D^{\delta_0} u(s, X_s(\cdot)) = Du(s, X_s(\cdot))(\{0\})$

### Remark

If $X = W$ and $h \in \mathbb{D}^{1,2}$

- Forward integral equals Itô integral
- The representation coincides with Clark-Ocone formula
- $H_0 = \mathbb{E}[h]$.

# Methodology: two steps

- We will choose a functional $u : [0, T] \times B \longrightarrow \mathbb{R}$ which solves the infinite dimensional PDE (2) with final condition $H$.
- Using Itô formula we establish the representation form (3).

## Explicit examples

1. $H(\eta) = f(\eta(0))$ where $f : \mathbb{R} \to \mathbb{R}$ continuous and polynomial growth $\Rightarrow$ $u$ such that $D^2 u(t,\eta) \in \mathcal{D}_{0,0}$ and $D^{ac} u(t,\eta) \equiv 0$.
2. $H(\eta) = \left( \int_{-T}^{0} \eta(s) ds \right)^2 \Rightarrow u$ such that $D^2 u(t,\eta) \in (\mathcal{D}_0 \oplus L^2) \hat{\otimes}_h^2$ and $D^{ac}$ with bounded variation.
3. $H(\eta) = \int_{-T}^{0} \eta(s)^2 ds \Rightarrow u$ such that $D^2 u(t,\eta) \in Diag \oplus \mathcal{D}_{0,0}$ and $D^{ac}$ not of bounded variation, we use $D^{\perp}$!

# A first example

1)

$$H(\eta) = \left( \int_{-T}^{0} \eta(s) ds \right)^2 \quad \Rightarrow \quad h := H\left(X_T(\cdot)\right) = \left( \int_0^T X_t dt \right)^2$$

We have

$$u(t,\eta) := \left( \int_{-T}^{0} \eta(s) ds + \eta(0)(T - t) \right)^2 + \frac{(T-t)^3}{3}$$

solves (2) and $h$ has representation (3).

$$\partial_t u(t,\eta) = -2\eta(0)\left(\int_{-T}^{0}\eta(s)ds + \eta(0)(T-t)\right) - (T-t)^2$$

$$D_{dx}u(t,\eta) = 2\left(\int_{-T}^{0}\eta(s)ds + \eta(0)(T-t)\right)\cdot$$
$$\cdot\left(\mathbb{1}_{[-T,0]}(x)dx + (T-t)\delta_0(dx)\right)$$

$$\begin{aligned} D^2_{dx\,dy}\phi(t,\eta) &= 2\mathbb{1}_{[-T,0]^2}(x,y)dx\,dy+ \\ &+ 2(T-t)\mathbb{1}_{[-T,0]}(x)dx\,\delta_0(dy)+ \\ &+ 2(T-t)\delta_0(dx)\,\mathbb{1}_{[-T,0]}(y)dy+ \\ &+ 2(T-t)^2\delta_0(dx)\,\delta_0(dy) \end{aligned}$$

$D^2u(t,\eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$ and $[X]_t = t$

# An interesting case

2)

$$H(\eta) = \int_{-T}^{0} \eta(s)^2 ds \quad \Rightarrow \quad h := H\left(X_T(\cdot)\right) = \int_0^T X_t^2 dt$$

$$u(t,\eta) := \int_{-T}^{0} \eta^2(s) ds + \eta(0)^2 (T-t) + \frac{(T-t)^2}{2}$$

solves (2) and $h$ has representation (3).

$$\partial_t u(t,\eta) = -\eta^2(0) - (T-t)\;;$$
$$D_{dx} u(t,\eta) = 2\eta(x)dx + 2\eta(0)(T-t)\,\delta_0(dx)$$
$$D^2_{dx\,dy}\phi(t,\eta) = 2\delta_y(dx)\,dy + 2(T-t)\delta_0(dx)\delta_0(dy) = 2\delta_x(dy)\,dx + 2(T$$

- $D^2 u(t,\eta) \in (Diag \oplus \mathcal{D}_{0,0})$ and $[X]_t = t$

# A general representation theorem 1/2

## Theorem

- i $H : B \longrightarrow \mathbb{R}$
- ii $u \in C^{1,2}\left([0,T[\times B\right) \cap C^0\left([0,T] \times B\right)$
- iii Some technical conditions on the existence of the integral.
- iv $(t,\eta) \mapsto D^2 u\,(t,\eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2 \oplus Diag$ continuously
- v $u$ solves the infinite dimensional PDE (2)

$$\begin{cases} \partial_t u(t,\eta) + \int_{]-t,0]} D^{\perp}_{dx} u(t,\eta)\, d^- \eta(x) + \frac{1}{2} \int_{D_t} D^2_{dx\,dy} u\,(t,\eta)\sigma^2(t,\eta(x)) = 0 \\ u(T,\eta) = H(\eta) \end{cases}$$

Sufficient conditions for iii: $D^{\perp}_{dx} u(t,\eta)$ admits a density $x \mapsto D^{ac}_x\, u(t,\eta)$ with bounded variation.

# A general representation theorem 2/2

## Theorem

Then

$$h = H(X_T(\cdot)) = u(T, X_T(\cdot))$$

has representation

$$h = H_0 + \int_0^T \xi_s d^- X_s$$

with

- $H_0 = u(0, X_0(\cdot))$
- $\xi_s = D^{\delta_0} u(s, X_s(\cdot))$.

The proof follows immediately applying the Itô's formula.

# Sufficient conditions on $H$

Sufficient conditions on $H$ to obtain $u$ which fulfills ii, iii, iv and v.

When $X$ is general process such that $[X]_t = \int_0^t \sigma^2(s, X_s) ds$.

1. $H$ has a smooth Fréchet dependence on $L^2([-T, 0])$.
2. $h := H(X_T(\cdot)) = f\left(\int_0^T \varphi_1(s) d^- X_s, \ldots, \int_0^T \varphi_n(s) d^- X_s\right)$.
   If $(\varphi_i) \in C^2([0, T]; \mathbb{R})$ for $i = 1, \ldots, n$ and some non degeneracy condition
   - If $\sigma = const \neq 0$ and $f : \mathbb{R}^n \to \mathbb{R}$ continuous and with linear growth.
   - If $\sigma \equiv 0$ and $f \in C^2(\mathbb{R}^n)$.

**Remark**

Weaker sufficient conditions on $H$ when $X = W$ ($\sigma \equiv 1$) if Clark-Ocone formula does not apply.
For instance when $h \notin \mathbb{D}^{1,2}$, or $h \notin L^2(\Omega)$ (even not in $L^1(\Omega)$).

# Stability results involving window Dirichlet processes

Let $D$ a real continuous $(\mathcal{F}_t)-$Dirichlet process,

$$D = M + A,$$

- $D$ a real continuous $(\mathcal{F}_t)-$Dirichlet process, $D = M + A$,
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- $A$ a zero quadratic variation process with $A_0 = 0$.

# Time-homogeneous Stability Theorem

## Theorem

Let

- $F : B \longrightarrow \mathbb{R}$ be $C^1$ Fréchet
- $DF : B \longrightarrow \mathcal{D}_0 \oplus L^2$ continuously

Then $F(D(\cdot))$ is an $(\mathcal{F}_t)$-Dirichlet process with local martingale component equal to

$$\tilde{M}_{\cdot} = F\big(D_0(\cdot)\big) + \int_0^{\cdot} D^{\delta_0} F\big(D_s(\cdot)\big)\, dM_s$$

where $D^{\delta_0} F(\eta) := DF(\eta)(\{0\})$.

# Stability results involving window weak Dirichlet processes

- $D$ a finite quadratic variation $(\mathcal{F}_t)-$weak Dirichlet process

$$D = M + A$$

- $M$ is the local martingale

# Stability Theorem

## Theorem

Let

- $F : [0, T] \times B \longrightarrow \mathbb{R}$ be $C^{0,1}$ Fréchet such that
- $DF : [0, T] \times B \longrightarrow \mathcal{D}_0 \oplus L^2$ continuously

Then $F(\cdot, D_\cdot(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part

$$\tilde{M}_t^F = F(0, D_0(\cdot)) + \int_0^t D^{\delta_0} F(s, D_s(\cdot)) dM_s \ .$$

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Thank you for your attention!!!

# A stochastic flow

## Definition

For $0 < s < t < T$ and $\eta \in B$ the **stochastic flow** is defined

$$Y_t^{s,\eta}(x) = \begin{cases} \eta(x+t-s) & x \in [-T, s-t] \\ \eta(0) + W_t(x) - W_s & x \in [s-t, 0] \end{cases}$$

where $W$ standard Brownian motion.

## Remark

- $(Y_t^{s,\eta})_{0 \leq s \leq t \leq T, \eta \in B}$ is a $B$-valued random field
- $$Y_r^{s,\eta} = Y_r^{t, Y_t^{s,\eta}} \quad \text{for} \quad 0 < s < t < r < T$$

# Theorem

Let $H : L^2([-T,0]) \longrightarrow \mathbb{R}$

- $H \in C^3(L^2[-T,0])$ with $D^2H \in L^2([-T,0]^2)$ and $D^3H$ polynomial growth
- $DH(\eta) \in H^1([-T,0])$ and other technical assumptions

$$u(t,\eta) := \mathbb{E}\left[H(Y_T^{t,\eta})\right]$$

Then

- $u \in C^{1,2}([0,T] \times B)$
- $u$ solves (??)

## Theorem

Let

- $$H(\eta) := f\left(\int_{[-T,0]} \varphi_1(u+T)d\eta(u), \ldots, \int_{[-T,0]} \varphi_n(u+T)d\eta(u)\right)$$
- $f : \mathbb{R}^n \to \mathbb{R}$ continuous and with linear growth and
- $(\varphi_i) \in C^2([0,T];\mathbb{R})$
- Matrix $\Sigma_t := (\Sigma_t)_{i,j} = \left(\int_t^T \varphi_i(s)\varphi_j(s)ds\right)$, $t \in [0,T]$.

$$\det(\Sigma_t) > 0 \qquad\qquad \forall\, t \in ]0,T[$$

### Remark

$\Sigma_t$ is the Covariance matrix of Gaussian vector $G := \left(\int_t^T \varphi_1(s)dW_s, \ldots, \int_t^T \varphi_n(s)dW_s\right)$

# Theorem

$$u(t,\eta) := \Psi\left(t, \int_{[-t,0]} \varphi_1(s+t)d\eta(s), \ldots, \int_{[-t,0]} \varphi_n(s+t)d\eta(s)\right)$$

with

$$\Psi(t, y_1, \ldots, y_n) = \int_{\mathbb{R}^n} f(z_1, \ldots, z_n)\, p(t, z_1 - y_1, \ldots, z_n - y_n) dz_1 \cdots dz_n$$

and $p \in C^{3,\infty}([0,T] \times \mathbb{R}^n)$ density of Gaussian vector $G$
Then

- $u \in C^{1,2}([0,T[\times B) \cap C^0([0,T] \times C([-T,0]))$
- $u$ solves (??)

## Remark

If $X = W$ an analougous result is true with a weaker condition on $f$

Let

- $f$ polynomial growth

Then

- $u \in C^{1,2}([0, T[\times B)$
- $$h = u(0, W_0(\cdot)) + \underbrace{\int_0^T D^{\delta_0} u(s, W_s(\cdot)) d^- W_s}_{\text{improper forward integral}}$$
- $u(0, W_0(\cdot)) = \mathbb{E}[h]$
- $f$ Lipschitz then $D^{\delta_0} u(s, W_s(\cdot)) = \mathbb{E}\left[D_s^m h | \mathcal{F}_t\right]$ since $h \in \mathbb{D}^{1,2}$

# Theorem

$H : B \longrightarrow \mathbb{R}$

- $$H(\eta) = f\left(\int_{-T}^{0} \eta(s)ds\right)$$
- $f : \mathbb{R} \longrightarrow \mathbb{R}$ Borel subexponential (not necessarily continuous)
- $h = f\left(\int_0^T W_s ds\right) \in L^1(\Omega)$

$$u(t,\eta) = \int_{\mathbb{R}} f\left(\int_{-T}^{0} \eta(r)dr + \eta(0)(T-t) + x\right) p_\sigma(t,x)dx$$

with $\sigma_t = \sqrt{\frac{(T-t)^3}{3}}$

# Theorem

Then

- $u \in C^{1,2}([0,T[\times B)$
- $$h = u(0, W_0(\cdot)) + \underbrace{\int_0^T D^{\delta_0} u(s, W_s(\cdot)) d^- W_s}_{\text{improper forward integral}}$$
- $u(0, W_0(\cdot)) = \mathbb{E}[h]$

## Remark

Since $h \notin L^2(\Omega)$, a priori neither Clark-Ocone formula nor its extensions to Wiener distributions apply

# A toy model for $X$ real valued

Let $X$ such that $[X]_t = t$

A1 $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous and polynomial growth

A2 $v \in C^{1,2}([0,T[\times\mathbb{R}) \cap C^0([0,T] \times \mathbb{R})$ such that

$$\begin{cases} \partial_t v(t,x) + \frac{1}{2}\partial_{xx} v(t,x) = 0 \\ v(T,x) = f(x) \end{cases}$$

Then

$$h := f(X_T) = v(0,X_0) + \underbrace{\int_0^T \partial_x v(s,X_s) d^- X_s}_{\text{improper forward integral}}$$

Schoenmakers-Kloeden (1999), Coviello-Russo (2006), Bender-Sottinen-Valkeila (2008)

# Considerations about previous representation in toy model

- If $X_t = W_t + t\,G$, $G$ non-negative r.v. $\notin L^1(\Omega)$ and $f(x) = x$ then $h = f(X_T) \notin L^1(\Omega)$.
- If $X = W$,
  1. A1 $\Longrightarrow h = f(W_T) \in L^p(\Omega)$, with $p \geq 1$. not new...but...
  2. $\begin{cases} f \text{ subexponential} \\ f(W_T) \in L^1(\Omega) \end{cases} \Longrightarrow$

$$h := f(W_T) = v(0, W_0) + \underbrace{\int_0^T \partial_x v(t, W_t) d^- W_s}_{\text{improper forward integral}}$$

**Remark**

$f$ not necessarily continuous, $v \notin C^0([0,T] \times R)$

# A first motivating example

1)

$$H(\eta) = \left( \int_{-T}^{0} \eta(s) ds \right)^2$$

$$u(t, \eta) := \left( \int_{-T}^{0} \eta(s) ds + \eta(0)(T - t) \right)^2 + \frac{(T - t)^3}{3}$$

solves (2) and $h$ has representation (3).

$$\partial_t u(t,\eta) = -2\eta(0)\left(\int_{-T}^{0}\eta(s)ds + \eta(0)(T-t)\right) - (T-t)^2$$

$$D_{dx}u(t,\eta) = 2\left(\int_{-T}^{0}\eta(s)ds + \eta(0)(T-t)\right)\cdot$$

$$\cdot\left(\mathbb{1}_{[-T,0]}(x)dx + (T-t)\delta_0(dx)\right)$$

$$\begin{aligned} D^2_{dx\,dy}\phi(t,\eta) = {} & 2\mathbb{1}_{[-T,0]^2}(x,y)dx\,dy + \\ & + 2(T-t)\mathbb{1}_{[-T,0]}(x)dx\,\delta_0(dy) + \\ & + 2(T-t)\delta_0(dx)\,\mathbb{1}_{[-T,0]}(y)dy + \\ & + 2(T-t)^2\delta_0(dx)\,\delta_0(dy) \end{aligned}$$

$D^2u(t,\eta) \in (\mathcal{D}_0 \oplus L^2)\hat{\otimes}_h^2$ and $[X]_t = t$

## Remark

If $X = W$

- Forward integral equals Itô integral
- The representation coincides with Clark-Ocone formula
- $H_0 = \mathbb{E}[h]$.

# An interesting case

2)

$$H(\eta) = \int_{-T}^{0} \eta(s)^2 ds$$

$$u(t,\eta) := \int_{-T}^{0} \eta^2(s)ds + \eta(0)^2(T-t) + \frac{(T-t)^2}{2}$$

solves (2) and $h$ has representation (3).

$$\partial_t u(t,\eta) = -\eta^2(0) - (T-t)\,;$$
$$D_{dx} u(t,\eta) = 2\eta(x)dx + 2\eta(0)(T-t)\,\delta_0(dx)$$
$$D^2_{dx\,dy}\phi(t,\eta) = 2\delta_y(dx)\,dy + 2(T-t)\delta_0(dx)\delta_0(dy) = 2\delta_x(dy)\,dx + 2(T$$

- $D^2u(t,\eta) \in (Diag \oplus \mathcal{D}_{0,0})$ and $[X]_t = t$
- $D^{ac}$ is not of bounded variation

# Stability result for $\mathbb{R}^n$ valued processes

In the finite dimensional case it holds.

**Theorem**

Let $X$ be a $\mathbb{R}^n$-valued process having all its mutual covariations $([X^*, X]_t)_{1 \le i,j \le n} = [X^i, X^j]_t$ and $F$, $G \in C^1(\mathbb{R}^n)$. Then the covariation $[F(X), G(X)]$ exists and is given by

$$[F(X), G(X)]_\cdot = \sum_{i,j=1}^{n} \int_0^\cdot \partial_i F(X) \partial_j G(X) d[X^i, X^j]$$

Setting $n = 2$, $F(x, y) = f(x)$, $G(x, y) = g(y)$, $f, g \in C^1(\mathbb{R})$ we have:

$$[f(X), g(Y)]_\cdot = \int_0^\cdot f'(X) g'(Y) d[X, Y]$$

# Stability result for $B$-valued processes

Previous results admit some generalizations in the infinite dimensional framework.

## Theorem

Let $X$ be a $B$-valued continuous stochastic process admitting a $\chi$-quadratic variation.
Let $F^i, F^j : B \longrightarrow \mathbb{R}$ be $C^1$ Fréchet such that for $i, j = 1, 2$

$$DF^i(\cdot) \otimes DF^j(\cdot) : B \times B \longrightarrow \chi \subset (B \hat{\otimes}_\pi B)^*$$
$$(x, y) \mapsto DF^i(x) \otimes DF^j(y) \quad \text{continuous}$$

Then $[F^i(X), F^j(X)]$ exists and it is given by

$$[F^i(X), F^j(X)]_\cdot = \int_0^\cdot \langle DF^i(X_s) \otimes DF^j(X_s), d\widetilde{[X]}_s \rangle$$

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Let $D$ a real continuous $(\mathcal{F}_t)-$Dirichlet process,

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Let

- $F : B \longrightarrow \mathbb{R}$ be $C^1$ Fréchet
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Then $F(D(\cdot))$ is an $(\mathcal{F}_t)$-Dirichlet process with local martingale component equal to

$$\tilde{M}_\cdot = F\big(D_0(\cdot)\big) + \int_0^\cdot D^{\delta_0} F\big(D_s(\cdot)\big)\, dM_s$$

where $D^{\delta_0} F(\eta) := DF(\eta)(\{0\})$.

# Stability results involving window weak Dirichlet processes

- $D$ a finite quadratic variation $(\mathcal{F}_t)-$weak Dirichlet process

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# Stability Theorem

## Theorem

Let

- $F : [0,T] \times B \longrightarrow \mathbb{R}$ be $C^{0,1}$ Fréchet such that
- $DF : [0,T] \times B \longrightarrow \mathcal{D}_0 \oplus L^2$ continuously

Then $F(\cdot, D_\cdot(\cdot))$ is an $(\mathcal{F}_t)$-weak Dirichlet process with martingale part

$$\tilde{M}_t^F = F(0, D_0(\cdot)) + \int_0^t D^{\delta_0} F(s, D_s(\cdot)) dM_s \ .$$