Contact process with aging

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Young Women In Probability
Bonn
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Contact Process

Definition

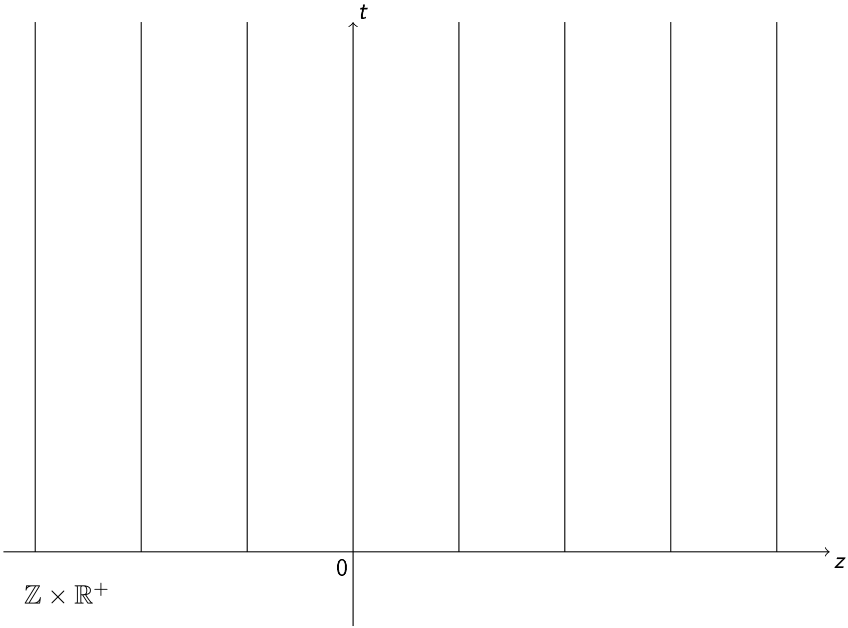
The contact process $\{\xi_t, t \geq 0\}$ is a continuous-time Markov process with values in $\{0,1\}^{\mathbb{Z}^d}$. Let $z \in \mathbb{Z}^d$:

- z is dead (or healthy) if $\xi_t(z) = 0$.
- z is alive (or infected) if $\xi_t(z) = 1$.

Rules of evolution :

- if a site is alive then it dies at rate 1 ;
- if a site is dead then it turns alive at rate λ times the number of its living neighbors.

The initial set ξ_0 is a finite set of $\mathbb{Z}^d$, we often take $\{0\}$.



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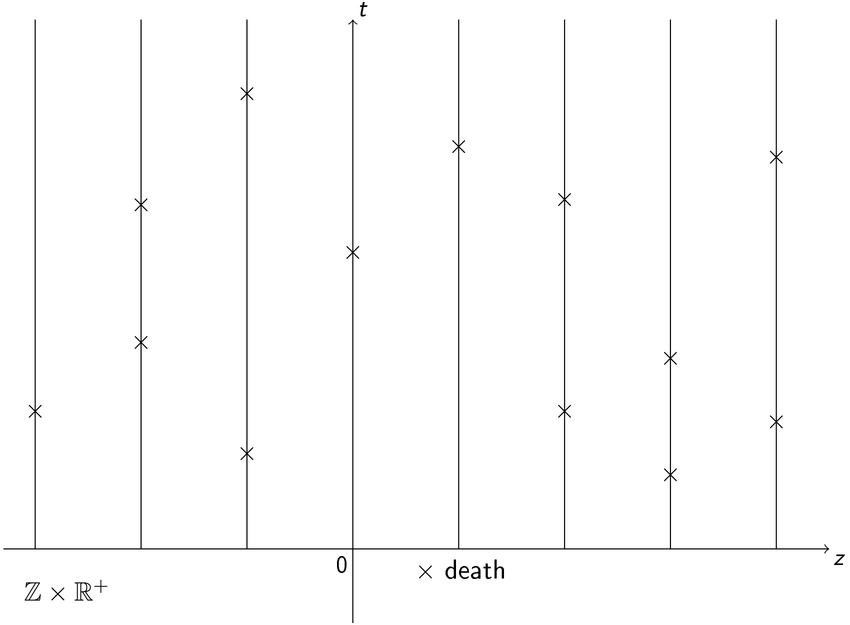
$\mathbb{Z} \times \mathbb{R}^+$



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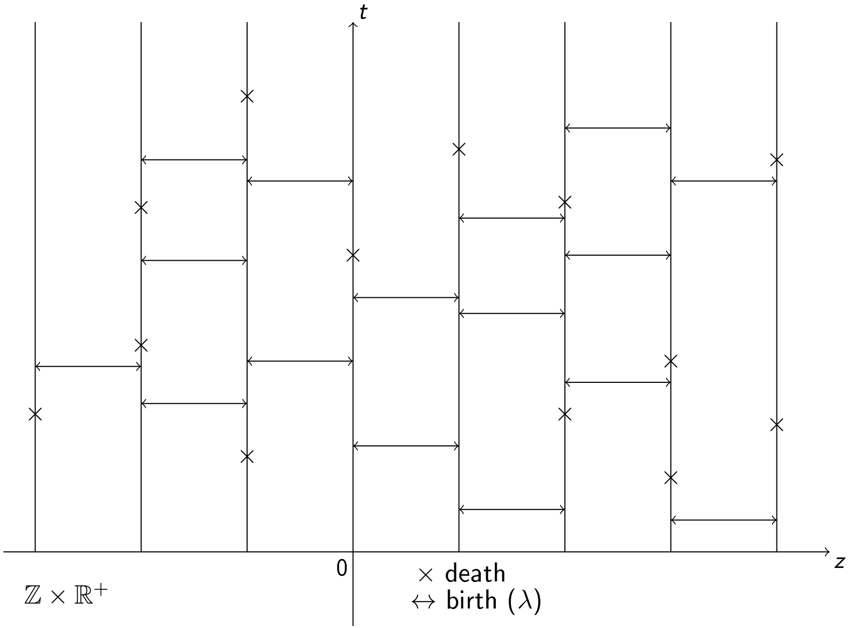

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↔ birth (λ)
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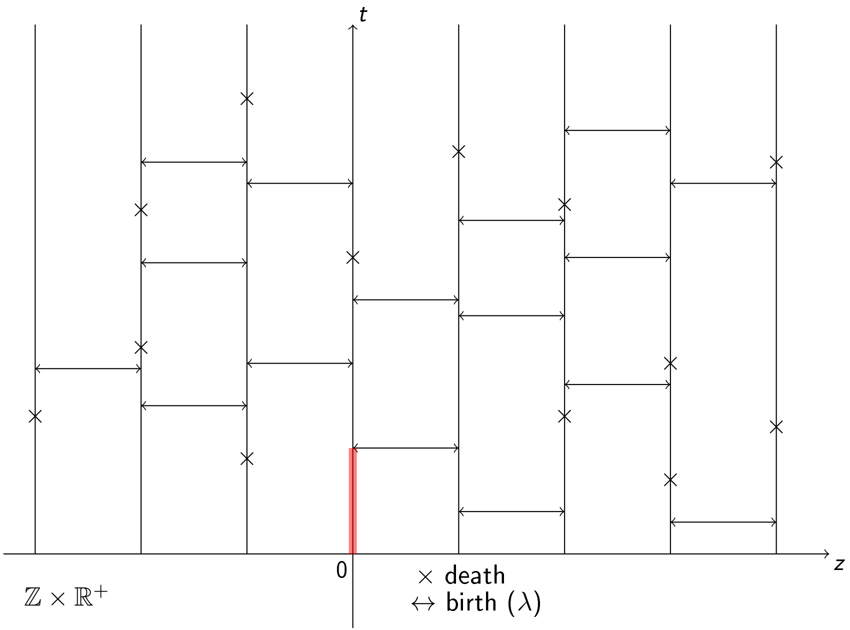

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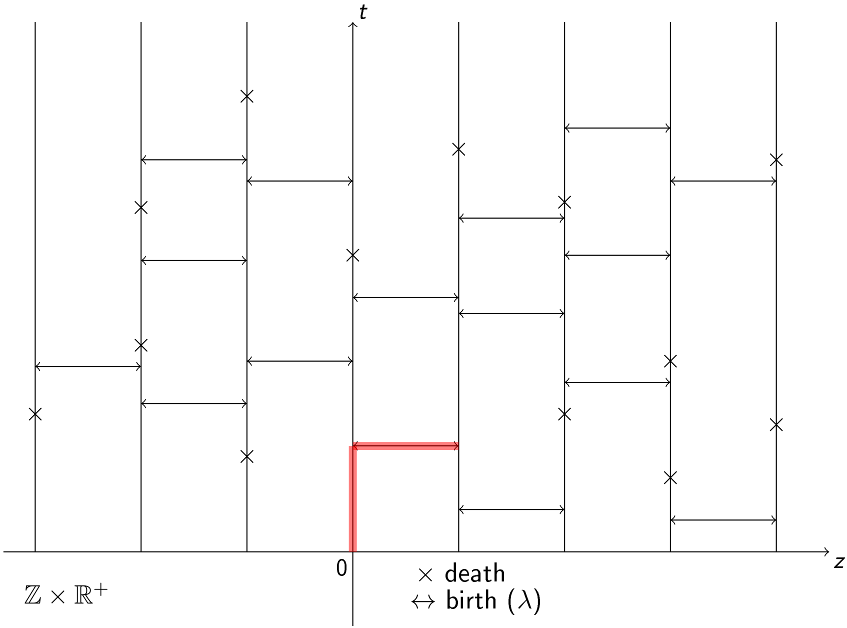

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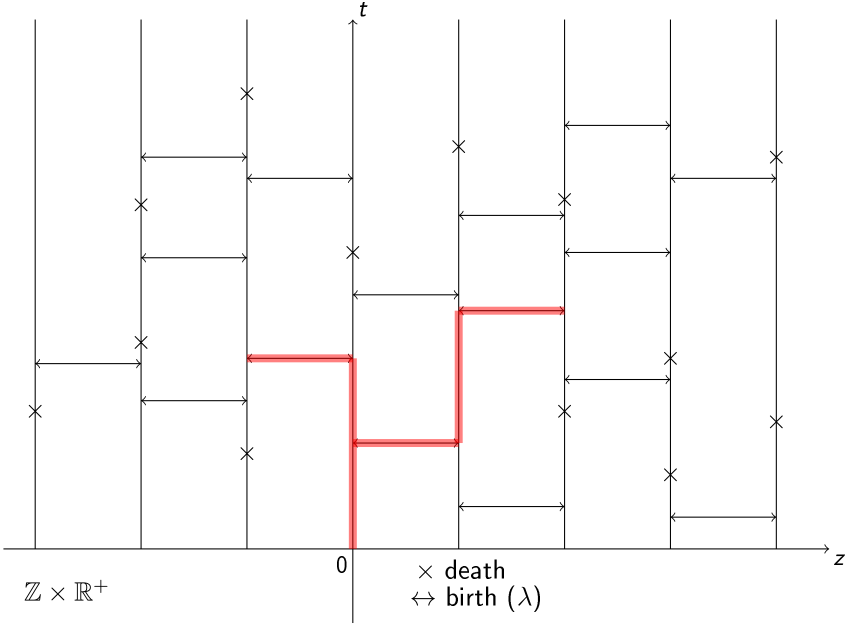

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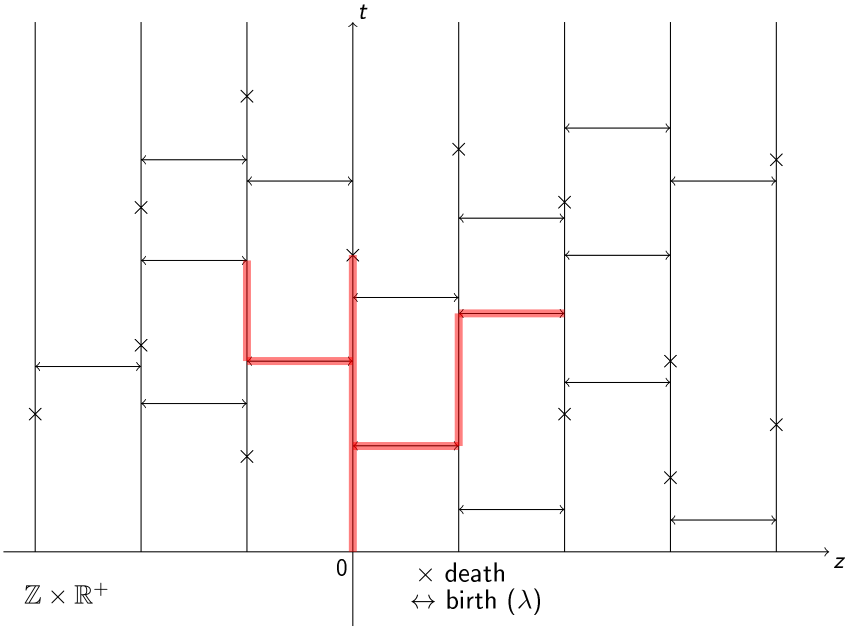

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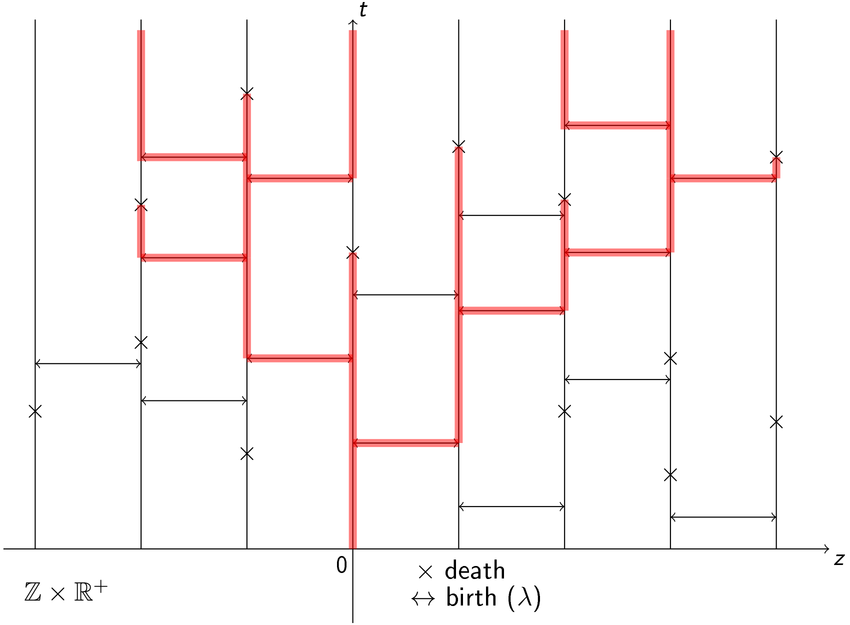
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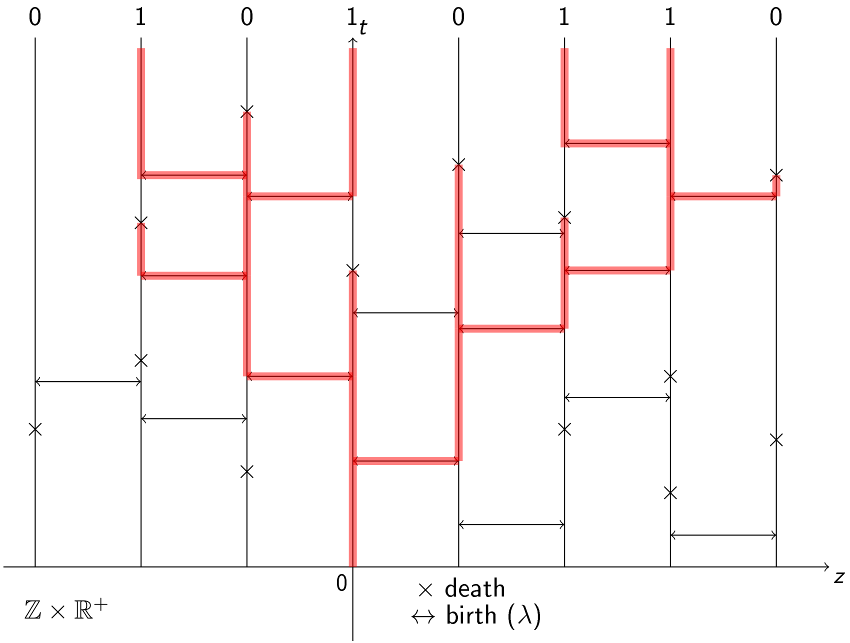
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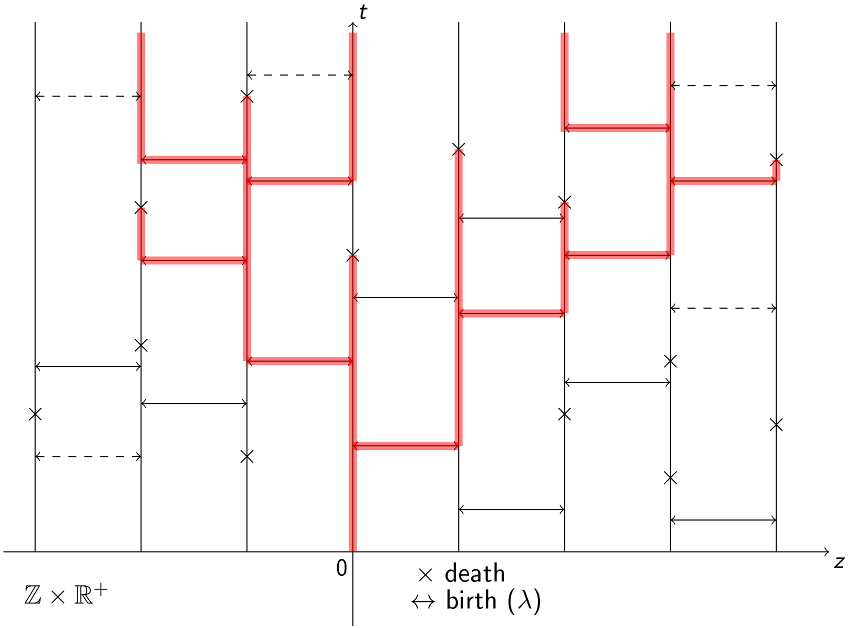

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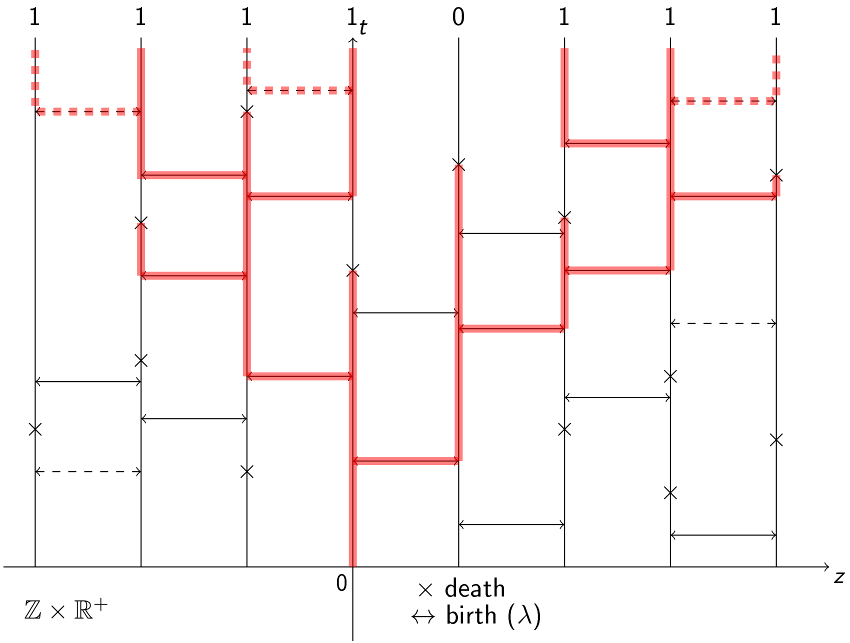

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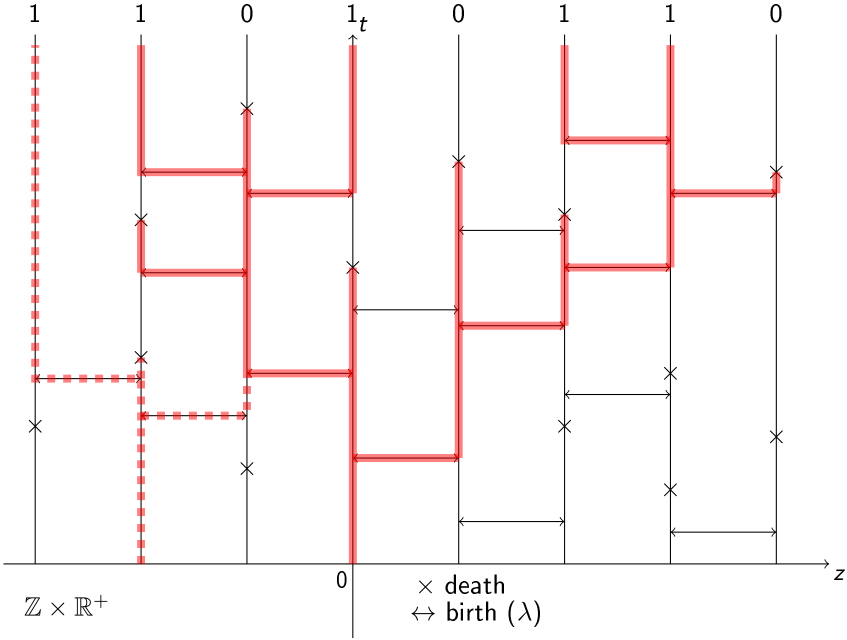

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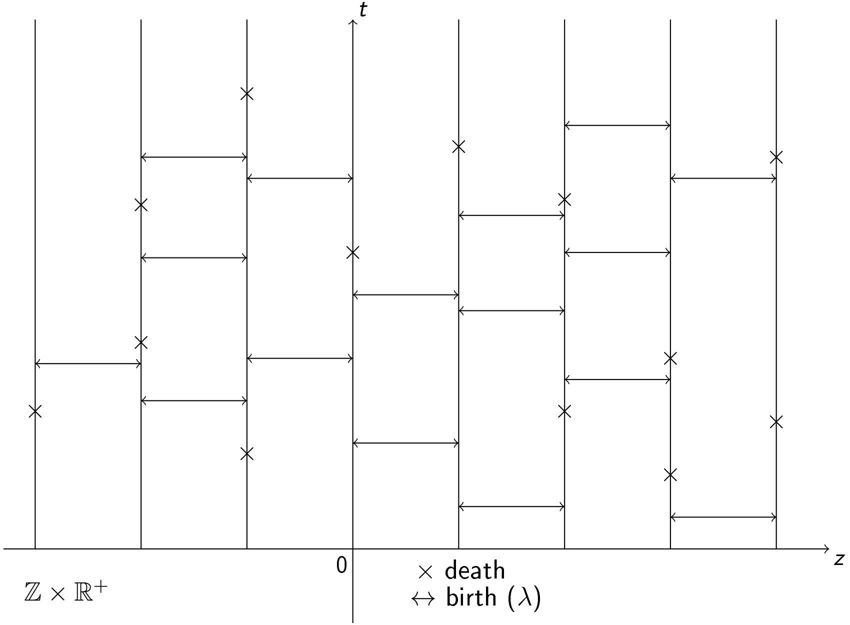

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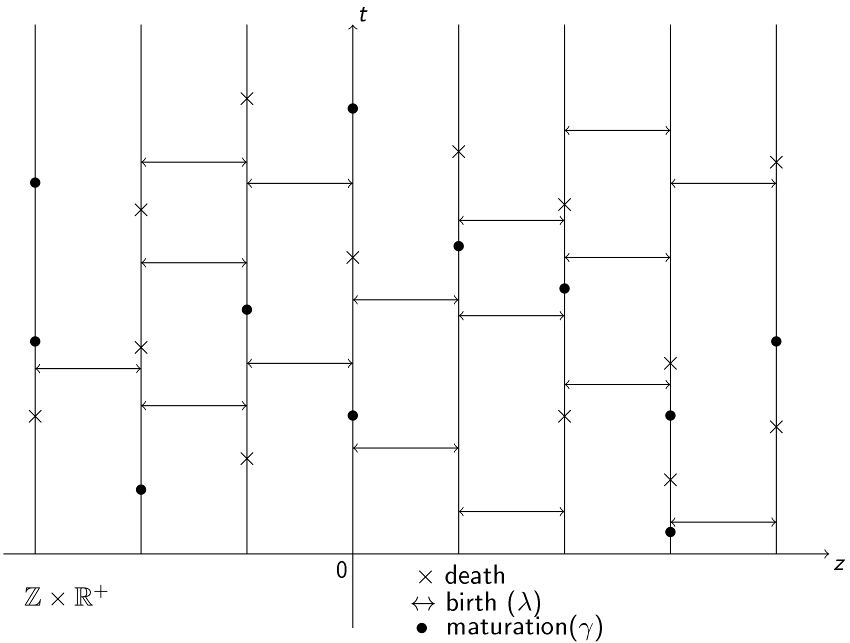

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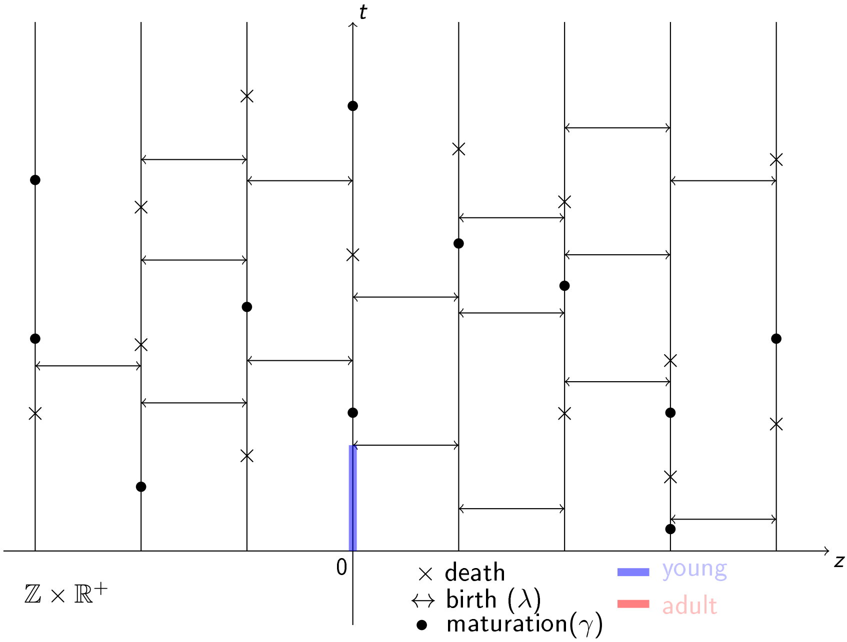

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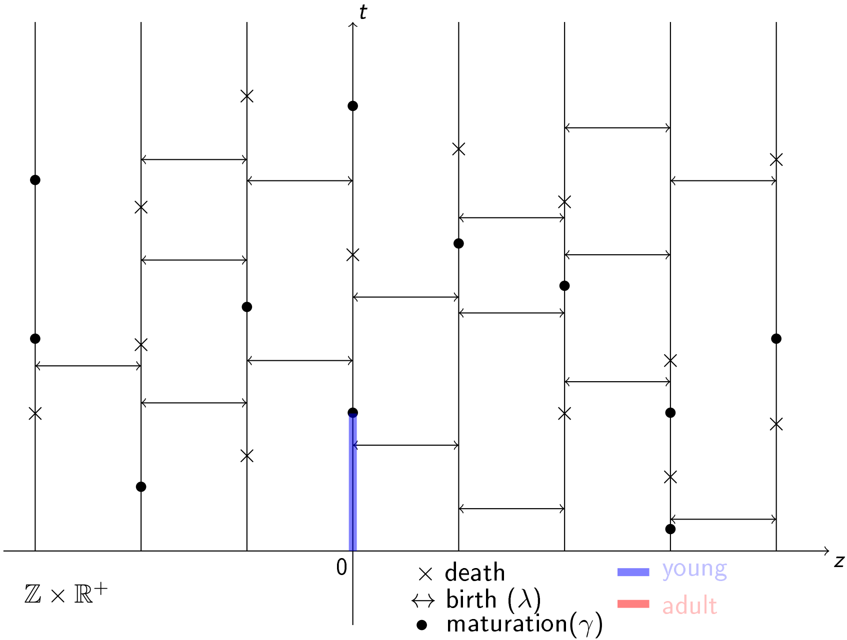

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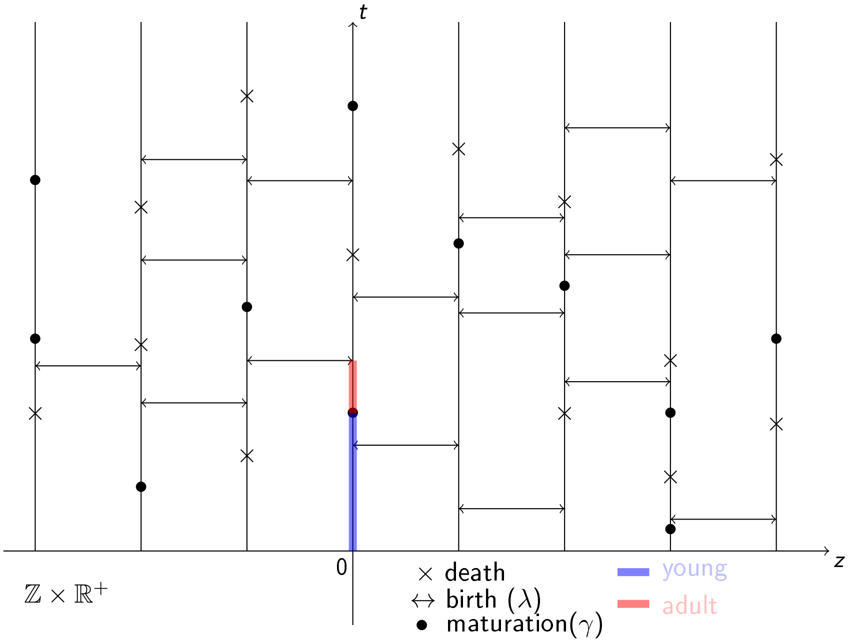

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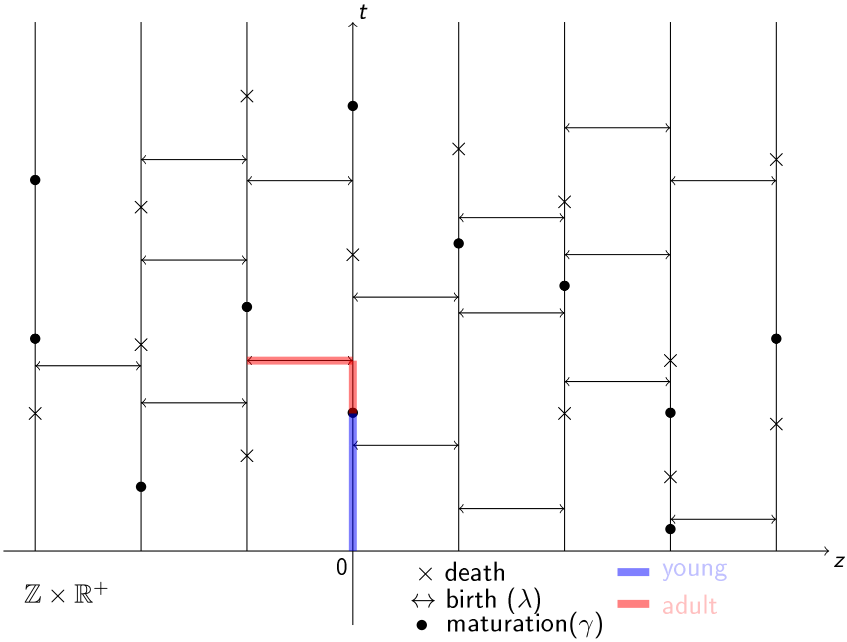

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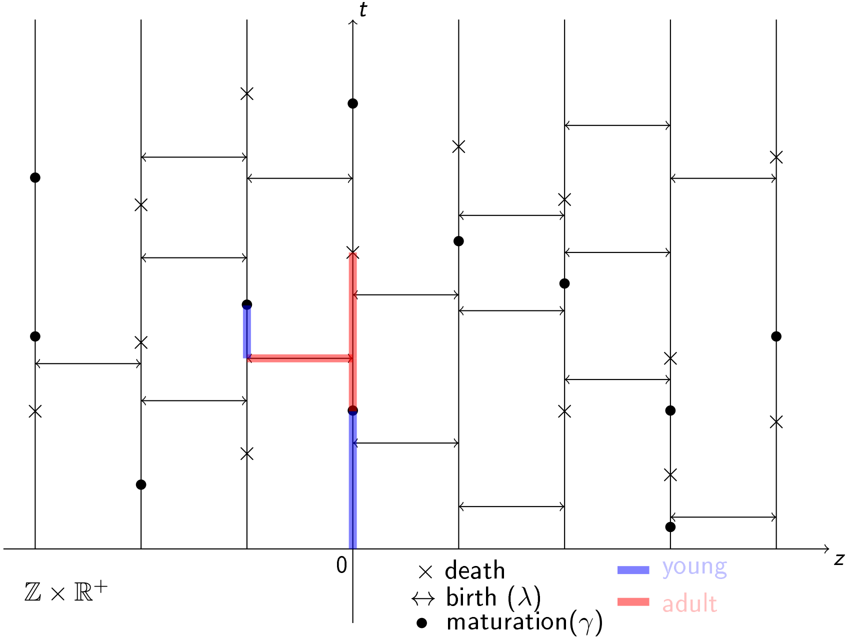

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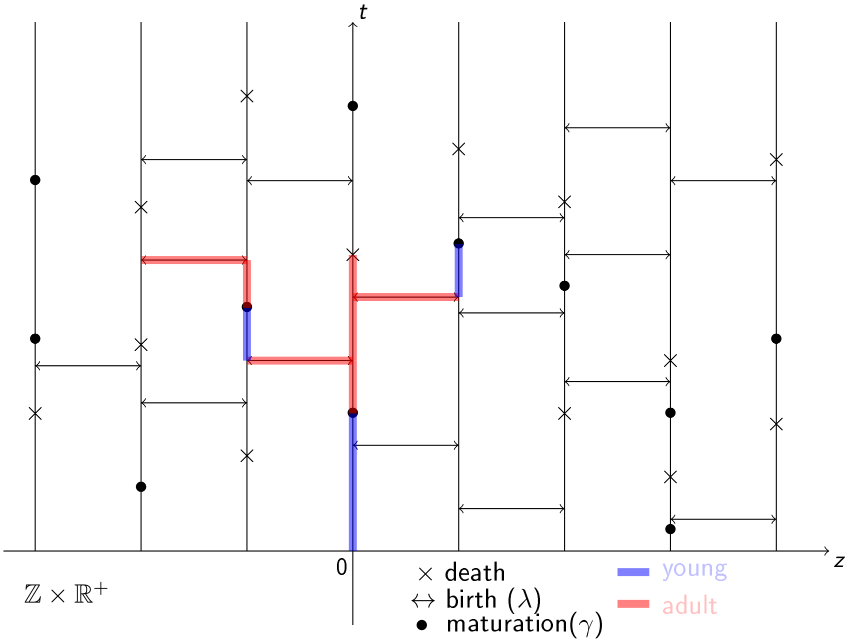

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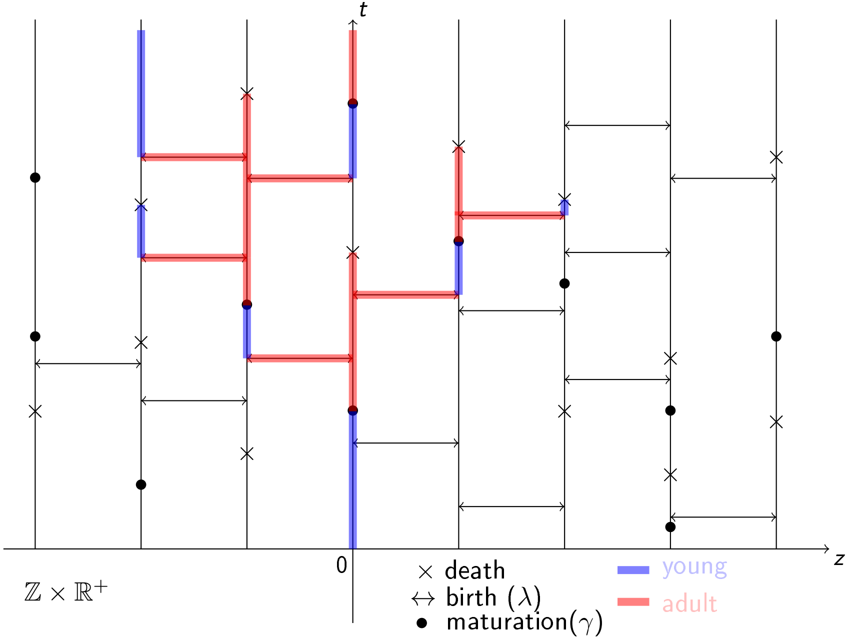
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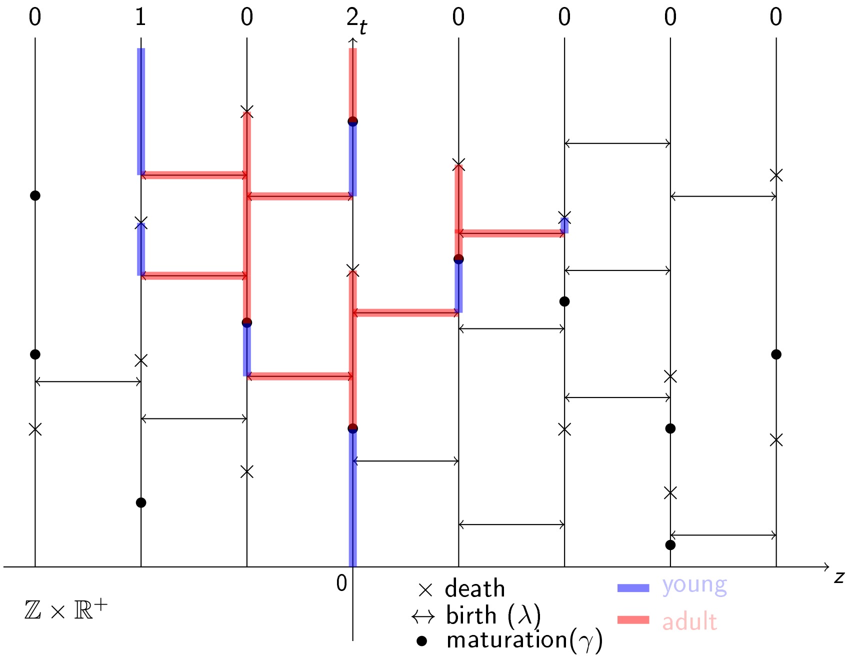

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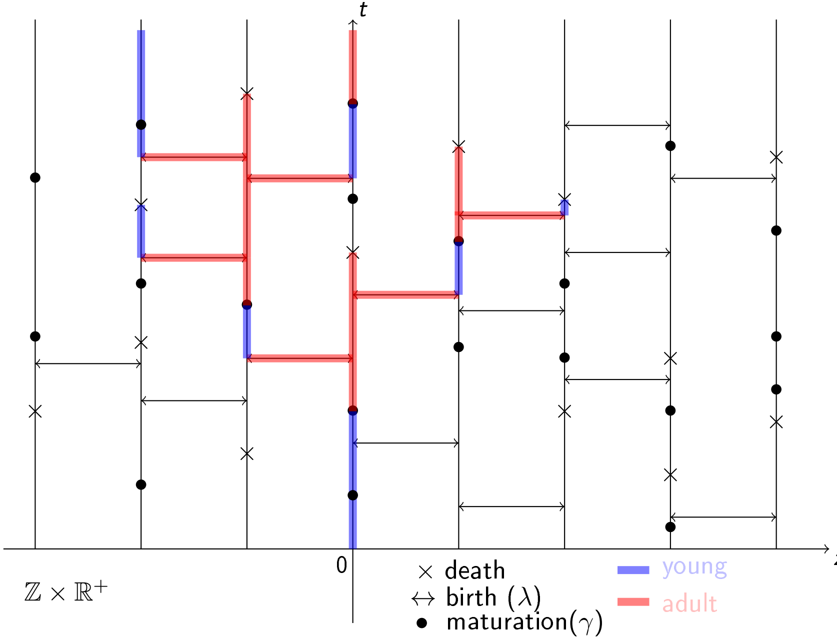

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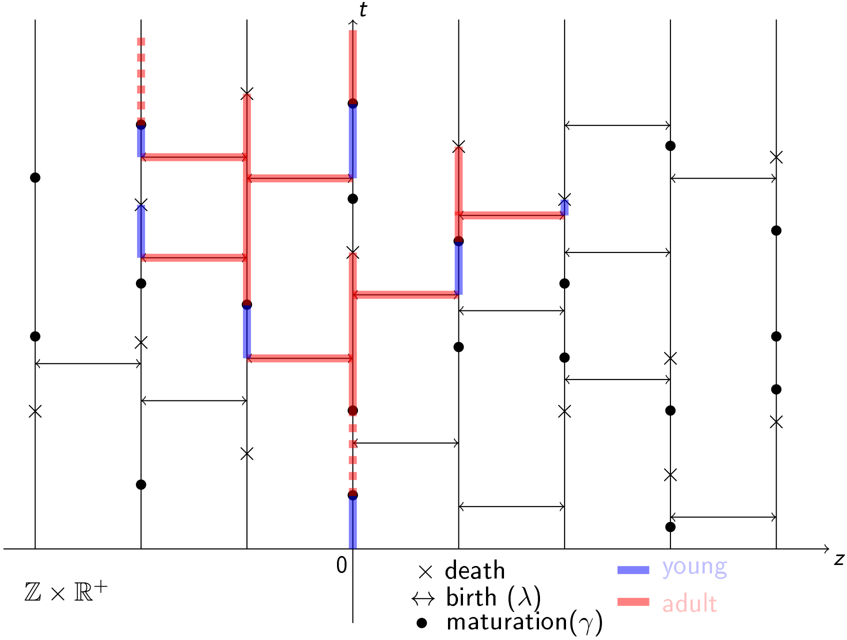

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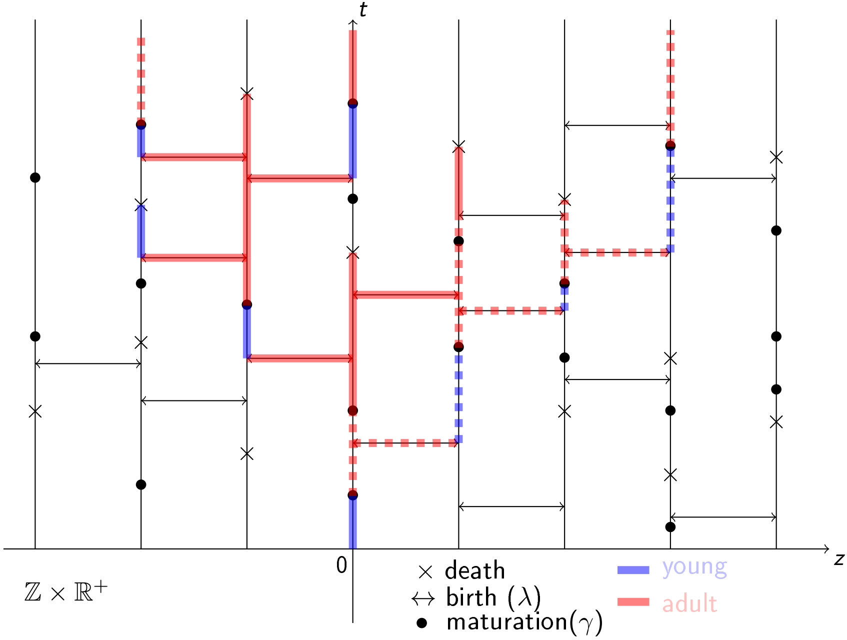

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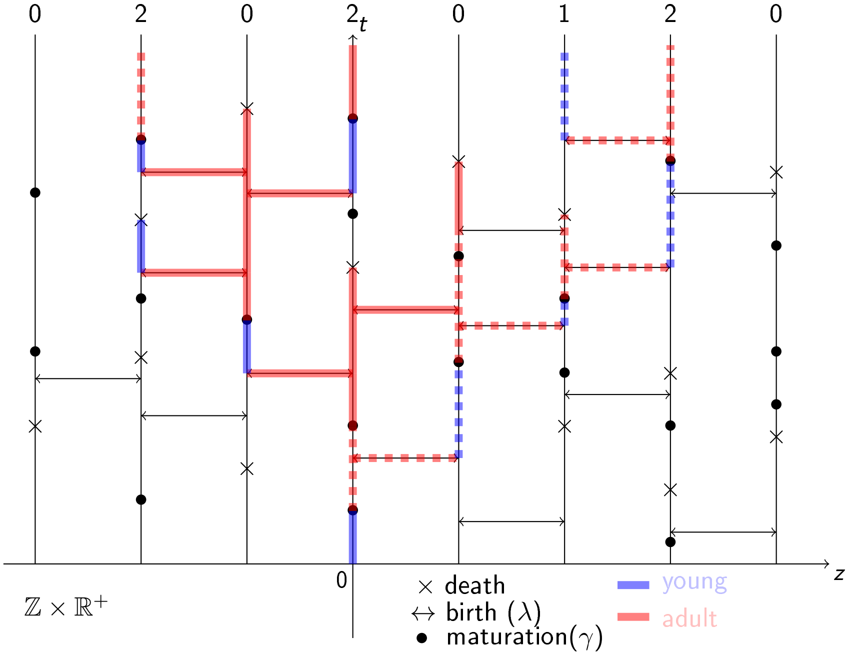

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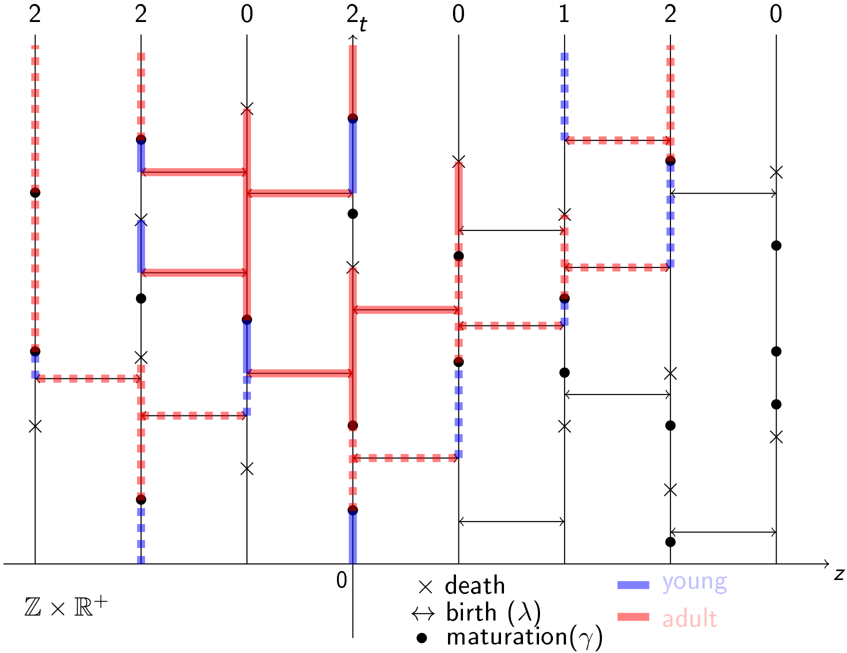
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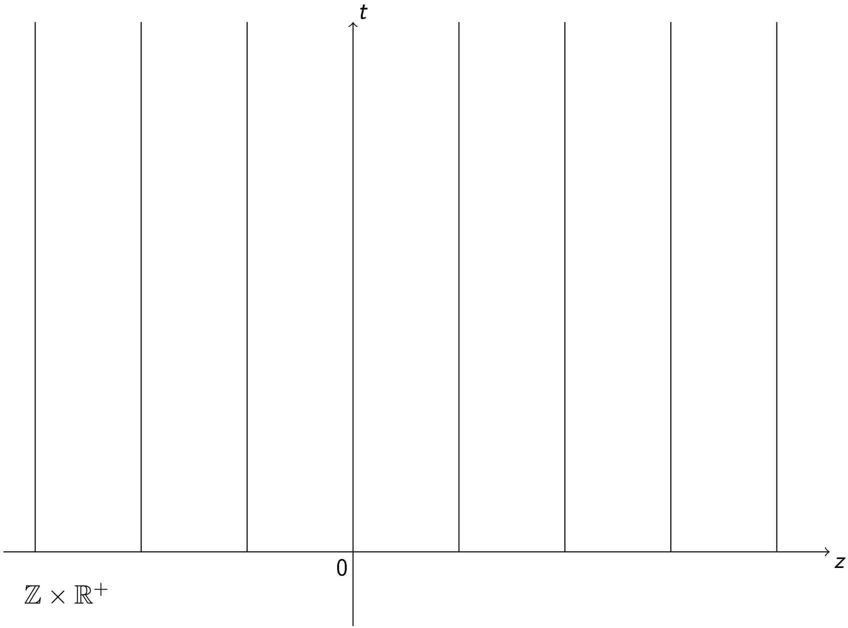

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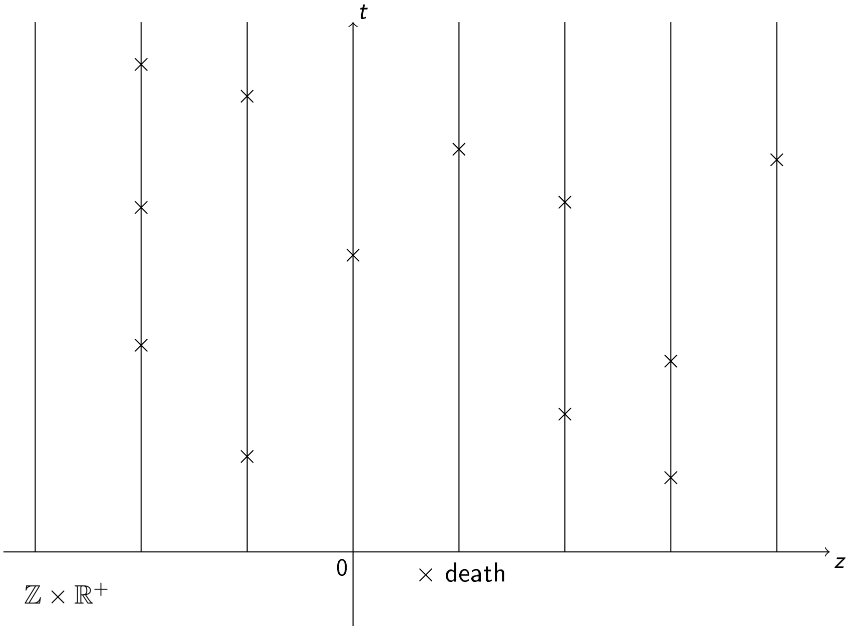

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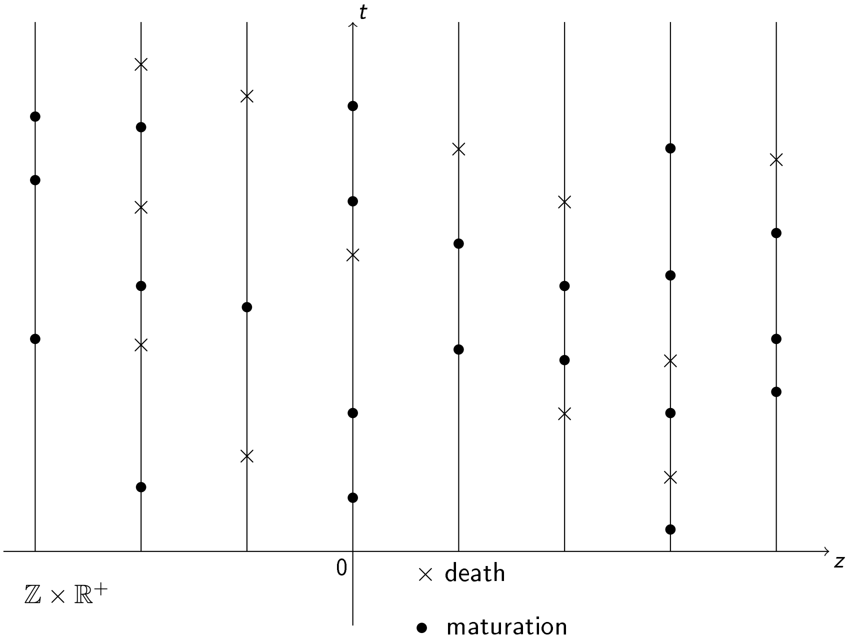

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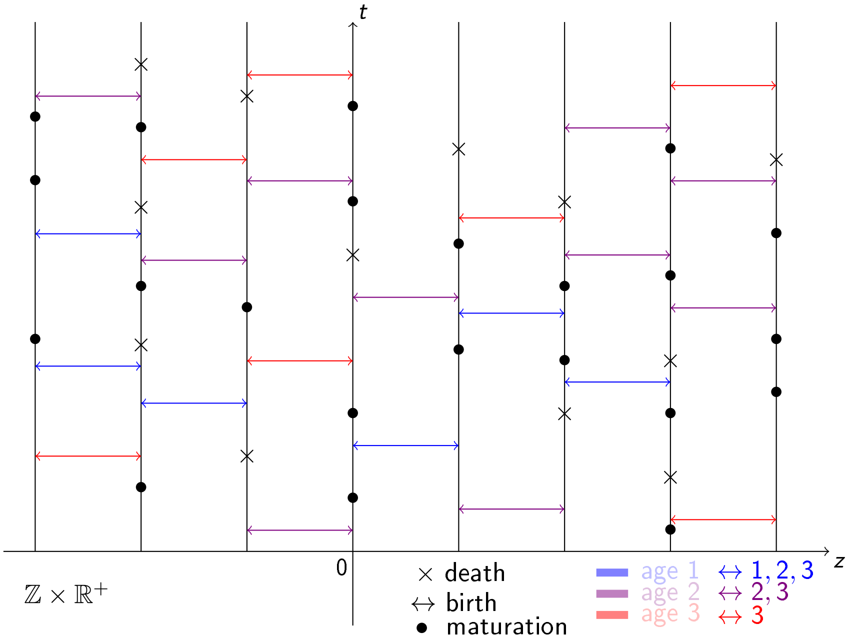

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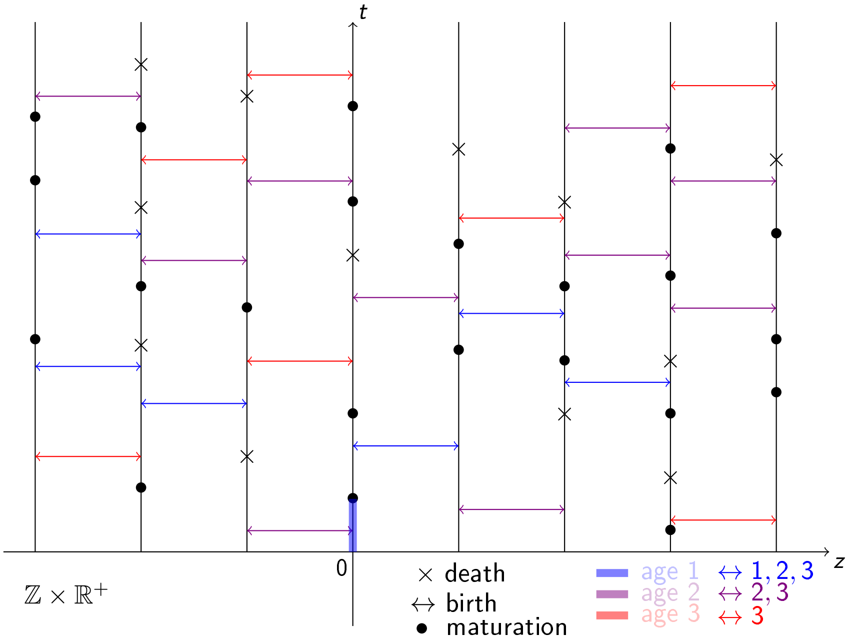
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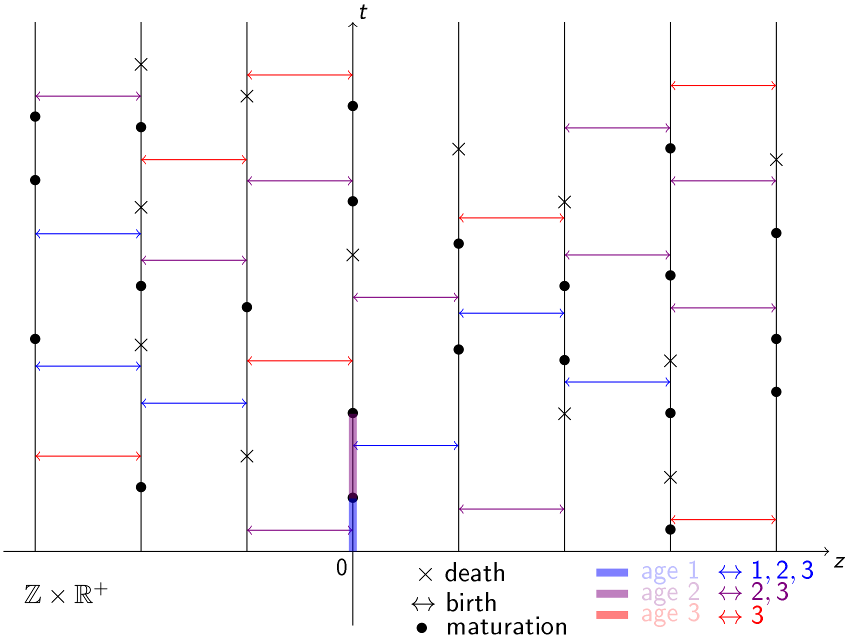

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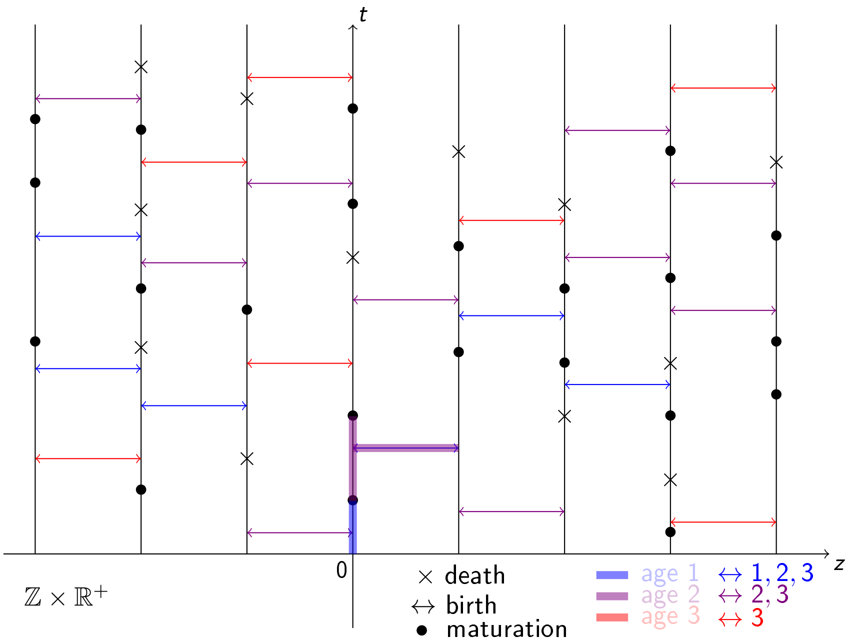

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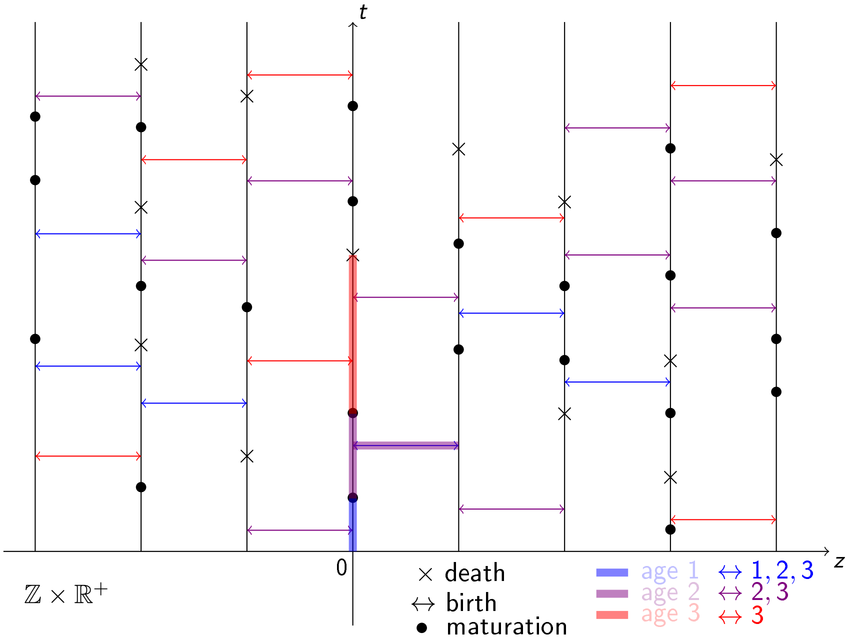

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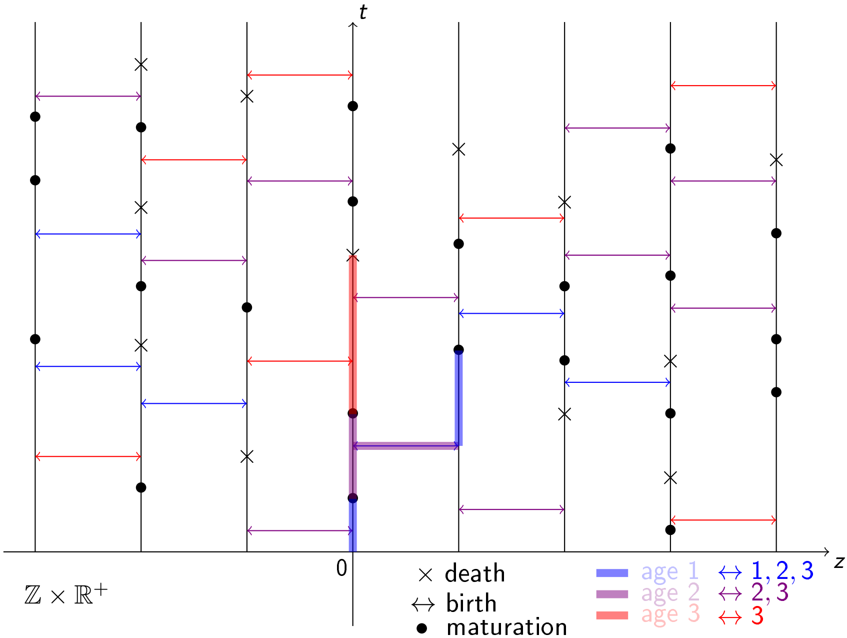

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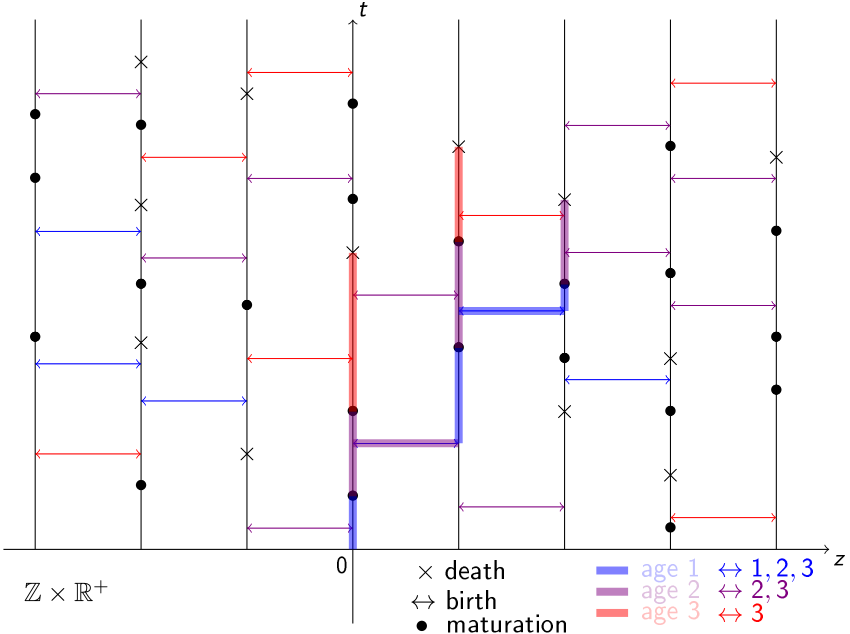

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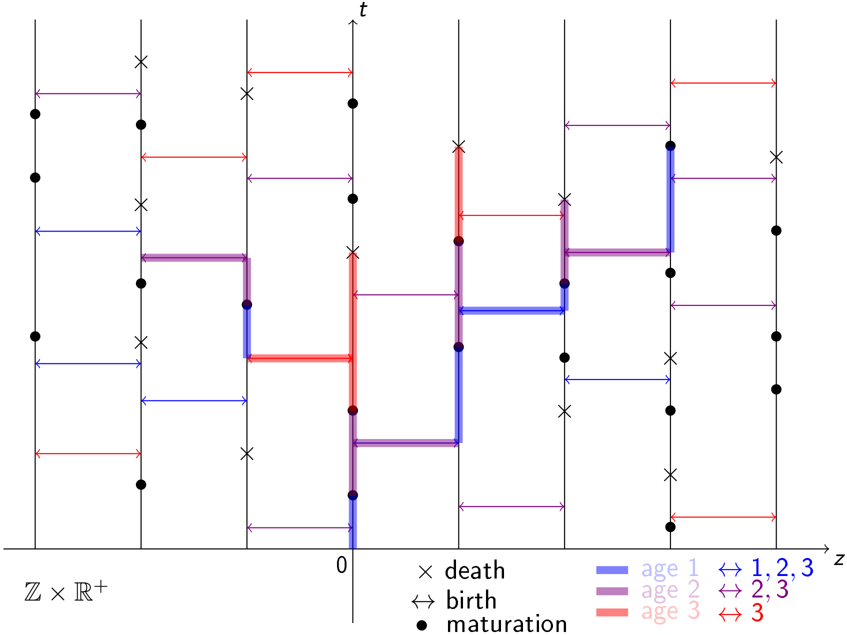

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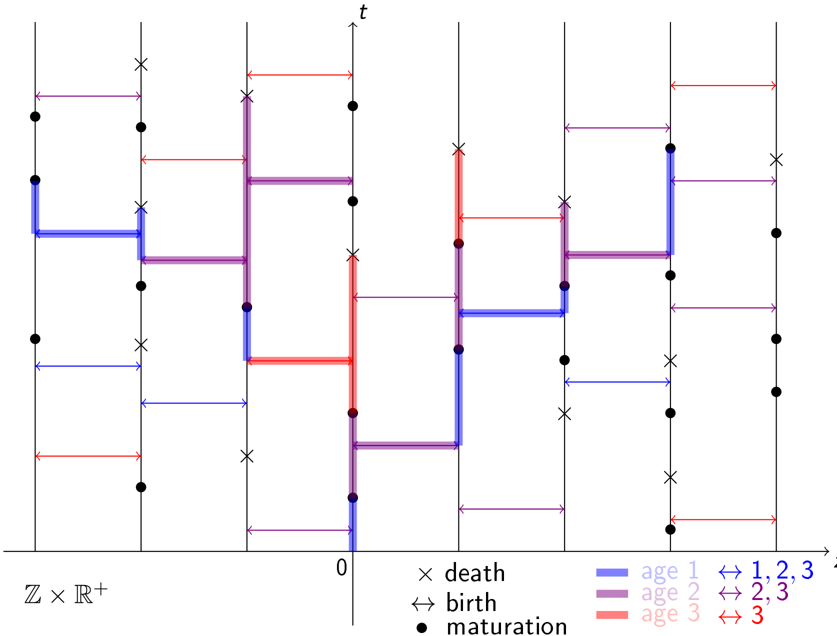

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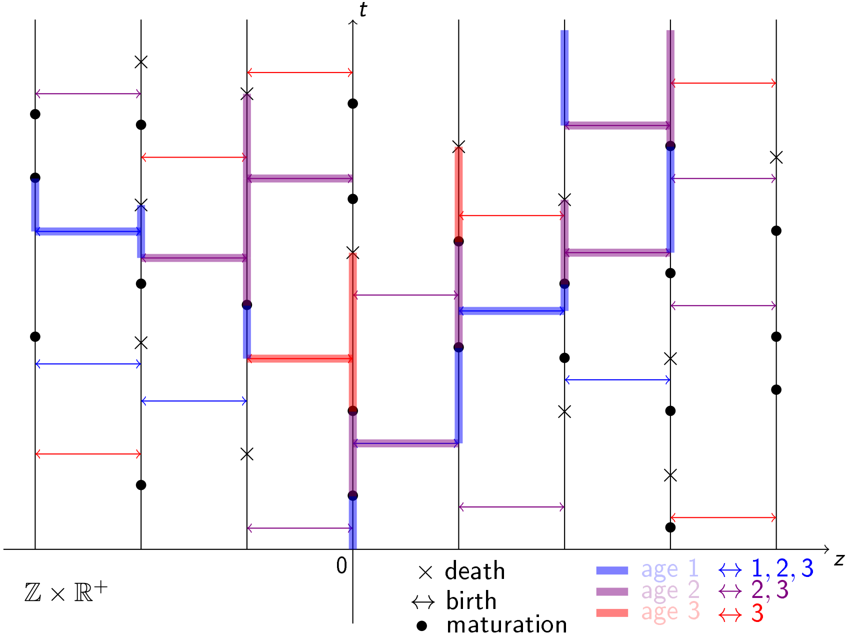

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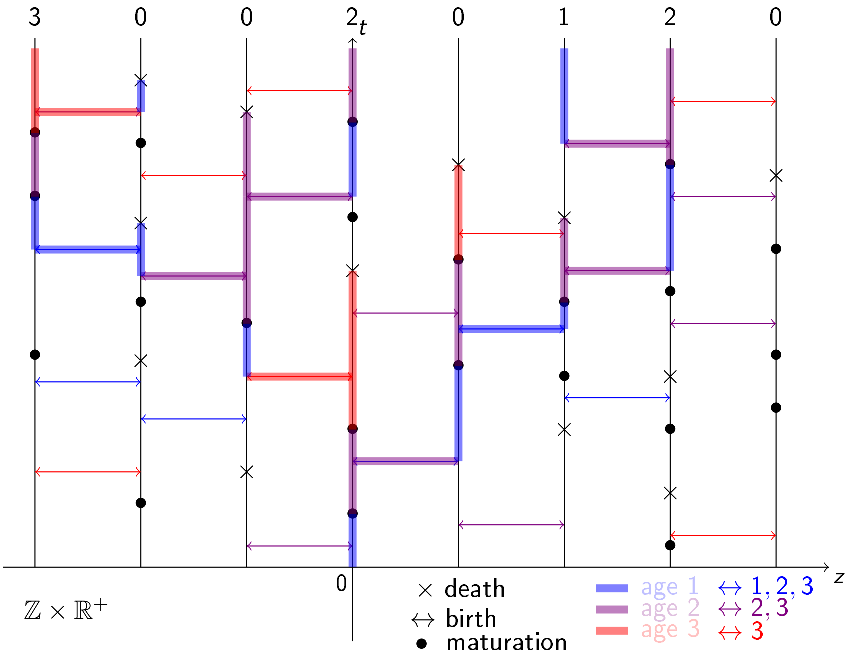

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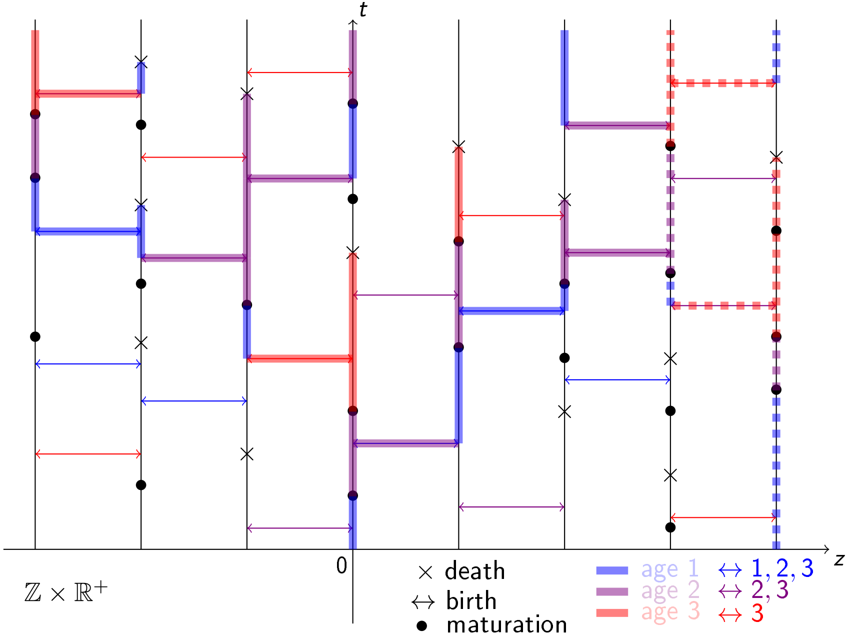

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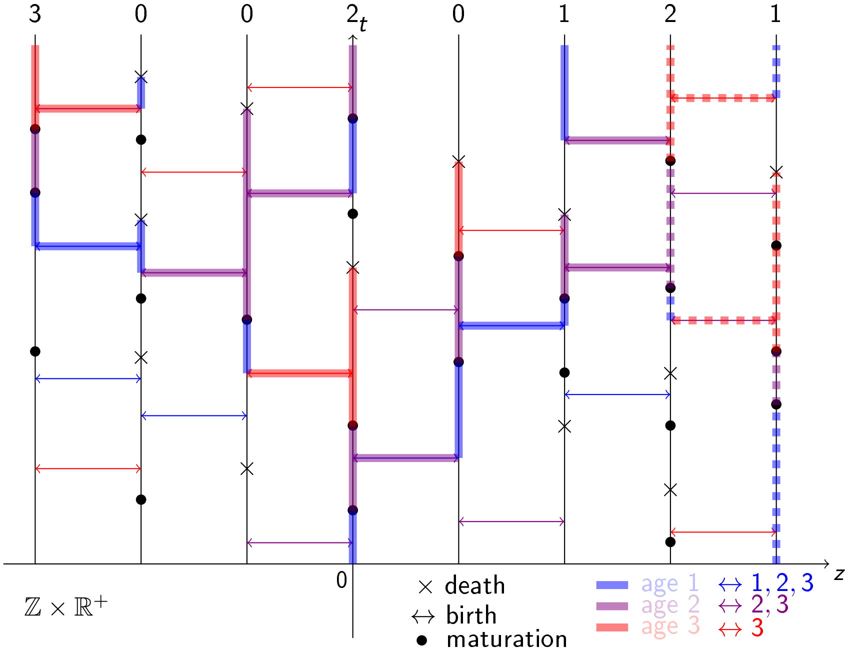
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Let $\Lambda = (\lambda_i)_{i\in\mathbb{N}}$ be the sequence of birth parameters :

1. $\forall i, \lambda_i \in \mathbb{R}^+$ and $\lambda_0 = 0$,
2. $(\lambda_i)_i$ is non decreasing,
3. $\lim_{i\to\infty} \lambda_i = \lambda_\infty < \infty$.

Let $\gamma > 0$ be the maturation parameter and $f : \mathbb{Z}^d \to \mathbb{N}$ with finite support.

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Let $\gamma > 0$ be the maturation parameter and $f : \mathbb{Z}^d \to \mathbb{N}$ with finite support.

Definition

A CPA $\{\xi_t^f, t \geq 0\}$ is a continuous-time Markov process with values in $\mathbb{N}^{\mathbb{Z}^d}$ and $\xi_0 = f$. Let $z \in \mathbb{Z}^d$ and $k \in \mathbb{N}^*$:

- z is dead if $\xi_t^f(z) = 0$,
- z is alive with age k if $\xi_t^f(z) = k$.

Evolution rules :

- a living site dies at rate 1 independently of its age,
- a dead site z turns alive at rate $\sum_{z', \|z'-z\|_1 = 1} \lambda_{\xi_t(z')}$ (with $\lambda_0 = 0$),
- each new offspring has age one,
- the transition from age n to age $n+1$ occurs at rate $\gamma > 0$, independently of its age.

For $t \geq 0$ we denote by :

$$A_t^f = \text{supp } \xi_t^f = \{x \in \mathbb{Z}^d ; \xi_t^f(x) \neq 0\},$$
$$= \text{set of living points at time } t.$$

Some properties

Let $f, g : \mathbb{Z}^d \to \mathbb{N}$ and $t > 0$.

- The CPA is **attractive** i.e $f \leq g \implies \xi_t^f \leq \xi_t^g$ and $A_t^f \leq A_t^g$;
- The CPA is **additive** i.e $\xi_t^{f \vee g} = \xi_t^f \vee \xi_t^g$;
- The CPA is **monotone** with respect to its parameters : non decreasing with respect to birth and maturation parameters

Goal : Prove an **asymptotic shape theorem** for $\bigcup_{s \leq t} A_t$.

Survival of contact process

Definition

We have **survival** if $\mathbb{P}(\forall t,\ \xi_t^{\{0\}} \neq 0) > 0.$

extinction if $\mathbb{P}(\forall t,\ \xi_t^{\{0\}} \neq 0) = 0.$

Critical value for the PC : $\lambda_c(\mathbb{Z}^d)$

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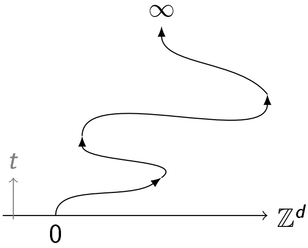

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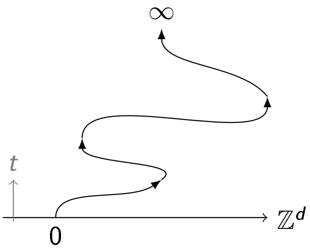

Critical value for the PC : $\lambda_c(\mathbb{Z}^d)$

Phase transition [Harris, 74]

There exists a critical value $\lambda_c \in (0, +\infty)$ such that :

- if $\lambda < \lambda_c$, then extinction,
- if $\lambda > \lambda_c$, then survival.

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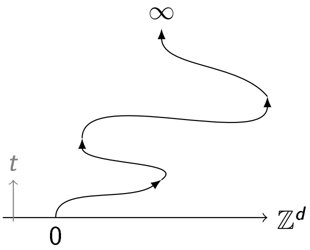
Definition

We have **survival** if $\mathbb{P}(\forall t,\ \xi_t^{\{0\}} \neq 0) > 0$.

extinction if $\mathbb{P}(\forall t,\ \xi_t^{\{0\}} \neq 0) = 0$.

Critical value for the PC : $\lambda_c(\mathbb{Z}^d)$



Phase transition [Harris, 74]

There exists a critical value $\lambda_c \in (0, +\infty)$ such that :

- if $\lambda < \lambda_c$, then extinction,
- if $\lambda > \lambda_c$, then survival.

At the critical point ? [Bezuidenhout and Grimmett, 91]

$$\lambda = \lambda_c, \text{ extinction.}$$

Krone introduces a similar quantity :

$$\lambda_c(\gamma) = \inf\left\{\lambda : \ \mathbb{P}_{\lambda,\gamma}\left(\forall t > 0, \ \xi_t^{\{0(2)\}} \neq \emptyset\right) > 0\right\}.$$

0(2) : the site 0 in adult state.

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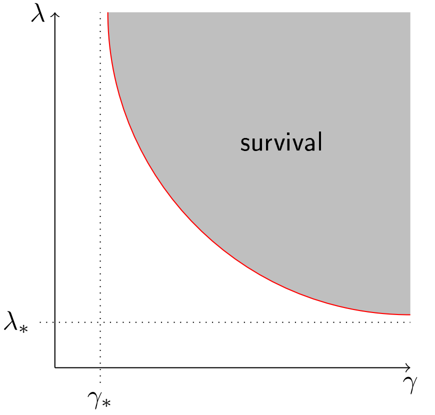

$\lambda_* = \lambda_c(PC)$. $\gamma_* > 0$ for $d = 1$ by Krone (1999) and $d \geq 1$ by Foxall (2014).

Survival region of CPA :

$$S_\gamma = \left\{ \Lambda \in \mathbb{R}^{\mathbb{N}},\ \text{non decreasing } \ / \ \mathbb{P}_{\Lambda,\gamma} \left(\forall t > 0,\ \xi_t \not\equiv 0 \right) > 0 \right\}.$$

- If $\Lambda \in S_\gamma$, then one has survival.
- If $\Lambda \notin S_\gamma$, then one has extinction.
- Let $\Lambda = (\lambda_i)_i$ and $\Lambda' = (\lambda'_i)_i$. If $\Lambda \in S_\gamma$ and $\forall i, \lambda_i \leq \lambda'_i$ then $\Lambda' \in S_\gamma$.

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If $\forall i, \lambda_i > \lambda_c(PC)$, then one has survival.

About the survival region...

For $m \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_m$ fixed, there exist λ_{m+1} and γ such that $\mathbb{P}_{\Lambda,\gamma}\left(\forall t > 0,\ \xi_t \not\equiv 0\right) > 0$.

To prove an asymptotic shape theorem...

Let Λ, γ such that $\mathbb{P}_{\Lambda,\gamma}\left(\forall t > 0,\ \xi_t^{\delta_0} \neq 0\right) > 0$. $\overline{\mathbb{P}}_{\lambda,\gamma}\left(\bullet\ |\forall t > 0,\ \xi_t^{\delta_0} \neq 0\right)$.

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We want to prove an **asymptotic shape theorem** for the CPA :

Theorem

There exists a norm μ on $\mathbb{R}^d$ such that for every $\epsilon > 0$ and every function $f : \mathbb{Z}^d \to \mathbb{N}$ with finite support, $\overline{\mathbb{P}}$-almost surely,

$$\forall t > 0,\ (1-\epsilon)B_\mu \subset \frac{\tilde{H}_t^f}{t} \subset (1+\epsilon)B_\mu,$$

where $\tilde{H}_t^f = \cup_{s \leq t} A_s^f + [0,1]^d$ and B_μ the unit ball of μ.

$\tilde{H}_t^f$ is (almost) the set of points born before t.

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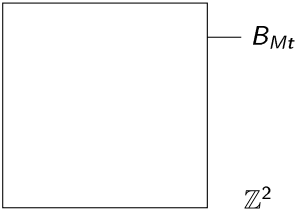
1. At most linear growth



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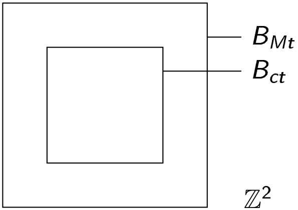
1. At most linear growth
2. At least linear growth



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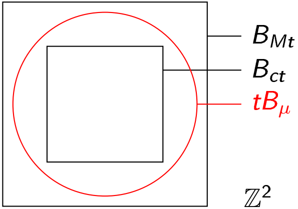
1. At most linear growth
2. At least linear growth
3. Exactly linear growth



1. At most linear growth

Let Λ, γ such that $\mathbb{P}_{\Lambda,\gamma}\left(\forall t > 0 \; \xi_t^{\delta_0} \neq 0\right) > 0$.

- $H_t^f = \cup_{s \leq t} A_s^f$ the set of points born before t
- Let $(\eta_t)_t$ a Richardson process (contact process without death)
- $B_R = \{y \in \mathbb{Z}^d : \|y\|_\infty \leq R\}$

There exist M, A, B such that for all $t > 0$, $\mathbb{P}(\eta_t \not\subseteq B_{Mt}) \leq A\exp(-Bt)$.

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Lemma

There exist A, B, M such that for every $f : \mathbb{Z}^d \to \mathbb{N}$ with finite support and every $t > 0$

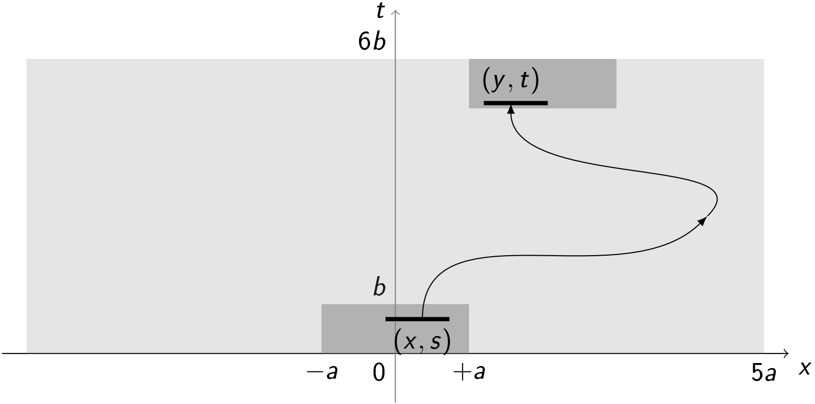
$$\mathbb{P}(H_t^f \not\subseteq B_{Mt}) \leq \mathbb{P}(\eta_t \not\subseteq B_{Mt}) \leq A\exp(-Bt).$$

2. At least linear growth

1. Block event
2. Percolation background
3. Expected estimates

1. Block event

If we have survival then $\forall \epsilon > 0\ \exists n, a, b$, such that the probability of the block event is at least $1 - \epsilon$.



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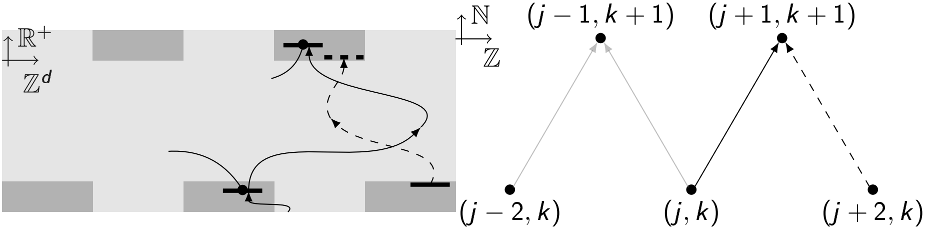

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3. Expected estimates

$$\tau^f = \inf\{t \geq 0 : \xi_t^f \equiv 0\} \text{ extinction time of CPA,}$$

Theorem

There exist A, B, C such that for all $t > 0$, $x \in \mathbb{Z}^d$ and $f : \mathbb{Z}^d \to \mathbb{N}$:

$$\mathbb{P}\big(t < \tau^f < \infty\big) \leq A \exp(-Bt),$$
$$\mathbb{P}(B_{ct} \not\subseteq H_t^f, \tau^f = \infty) \leq A \exp(-Bt).$$

3. Exactly linear growth $\to$ shape theorem

$$t^f(x) = \inf\{t \geq 0 : \xi_t^f(x) \neq 0\} \text{ hitting time of } x.$$
$$H_t^f = \{x \in \mathbb{Z}^d : t^f(x) \leq t\}.$$

- Let $x \in \mathbb{Z}^d$. We want to prove the convergence of $\frac{t(nx)}{n}$ when $n \to \infty$.
- Ergodic subadditive theorem of Kingman :

$$t((n+p)x) \leq t(nx) + t(px) \circ \text{space-time translation.}$$

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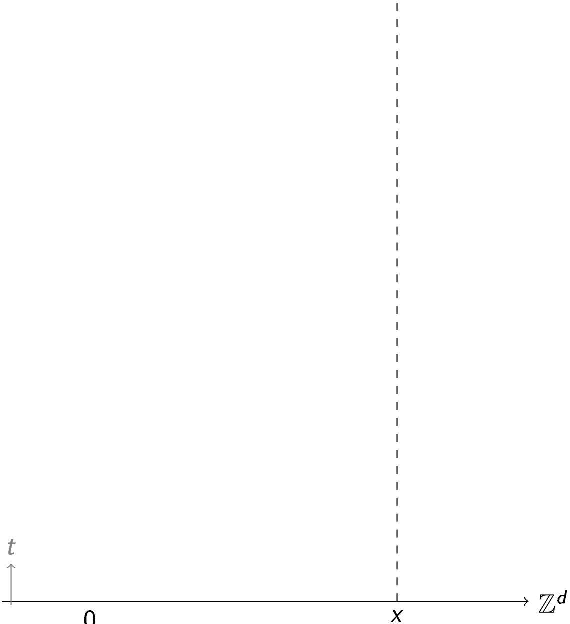
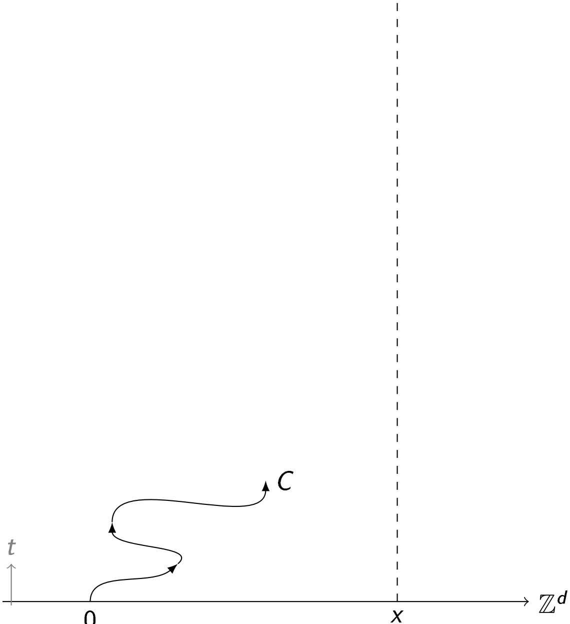
	$t(x)$	$t(x)$ under $\overline{\mathbb{P}}$
integrability	NO	YES
stationarity	YES	NO
subadditivity	YES	NO



t
$\mathbb{Z}^d$
0
x



C
t
0
x
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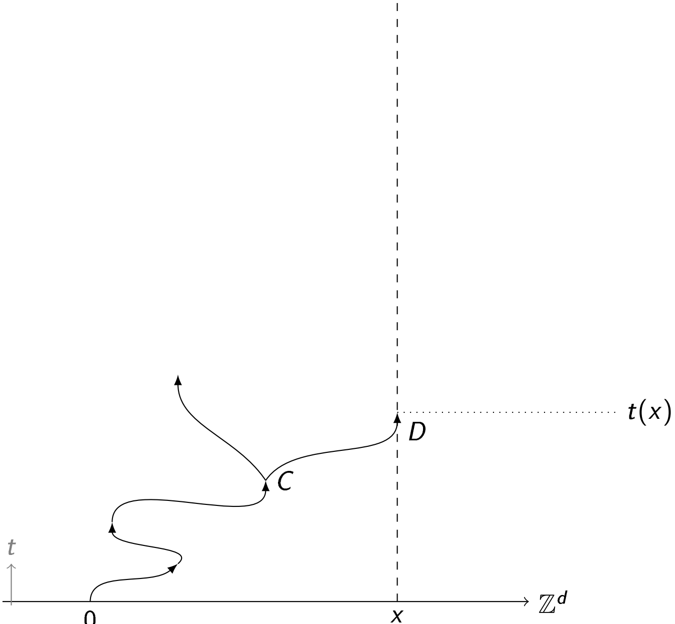
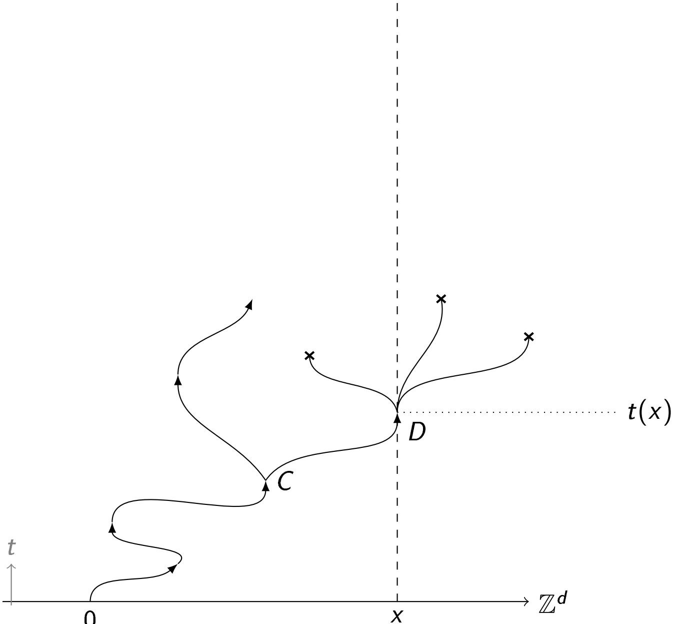


t
C
D
$t(x)$
0
x
$\mathbb{Z}^d$



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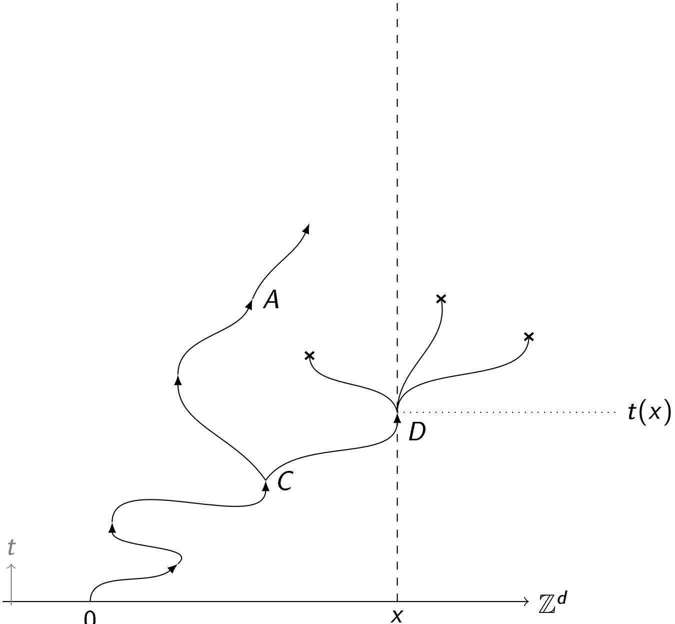
A
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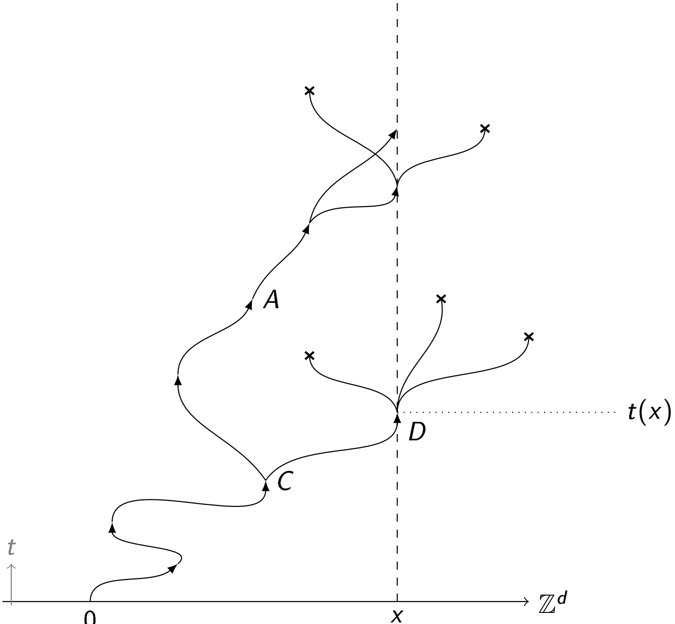

t
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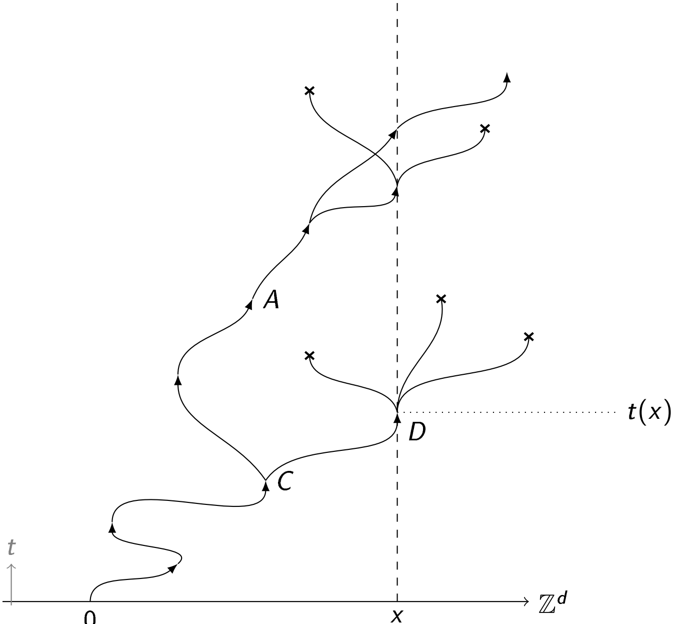

A
$t(x)$
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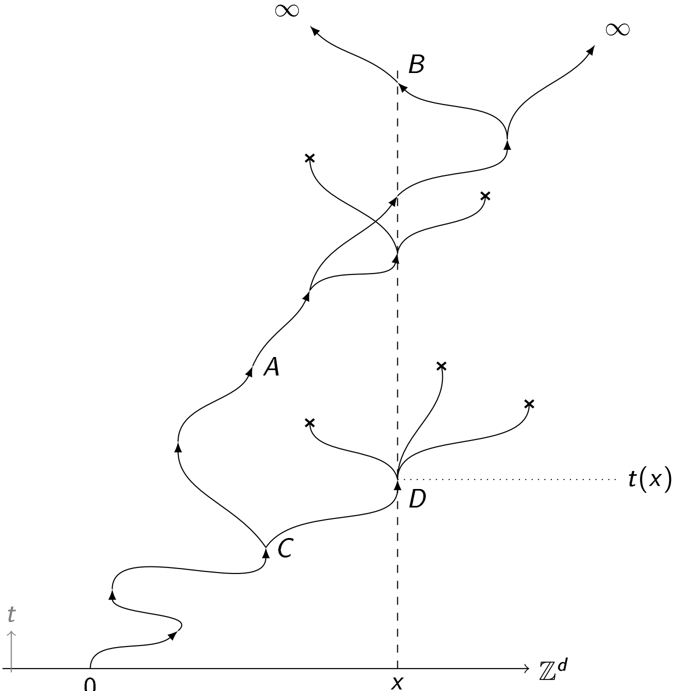

∞
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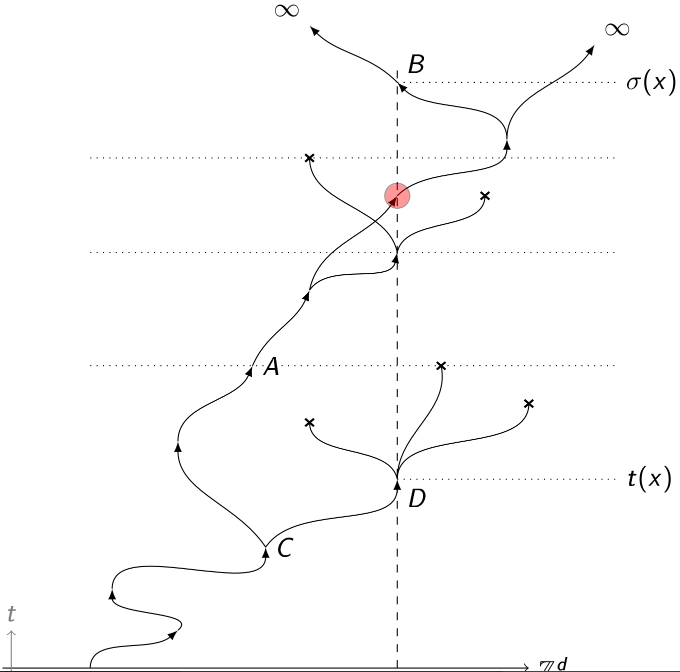
t(x)
D
C
t
0
x
$\mathbb{Z}^d$


∞
∞
B
σ(x)
A
t(x)
D
C
t
$\mathbb{Z}^d$

3. Exactly linear growth $\rightarrow$ shape theorem

	$t(x)$	$t(x)$ under $\overline{\mathbb{P}}$	σ under $\overline{\mathbb{P}}$
integrability	NO	YES	YES
stationarity	YES	NO	YES
(almost) subadditivity	YES	NO	YES

3. Exactly linear growth → shape theorem

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- σ satisfies also growth of linear order (as t).

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- directional convergence for σ thanks to the almost subadditive theorem of Kesten- Hammersley(74) :

$$\sigma((n+p)x) \leq \sigma(nx) + \sigma(px) \circ \text{space-time translation} + \text{some controlled quantity.}$$

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- « Uniform continuity » of σ $\rightarrow$ shape theorem for σ.

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- « Uniform continuity » of σ $\rightarrow$ shape theorem for σ.
- Control of the difference between t and σ $\rightarrow$ shape theorem for t.

Thanks for your attention