Construction and Properties of Continuous-state Branching Processes with Memory

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Galton-Watson process

$$\{Z(n)\}_{n\in\mathbb{N}}, \text{ with } Z(n) \in \mathbb{N}.$$

Definition

Let $\{Z_n\}_{n\in\mathbb{N}}$ be a Markov chain s.t.

$$\begin{cases} Z_0 > 0 \\ Z_n = \sum_{i=1}^{Z_{n-1}} \xi_i^{(n)} \text{ per } n = 1, 2, \dots \end{cases} \tag{1}$$

and $\xi_i^{(n)}$ are i.i.d. random variables in $\mathbb{N}$.

Branching property

Let $\{Z_n\}_{n\in\mathbb{N}}$, $\{Z_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{Z_n^{(2)}\}_{n\in\mathbb{N}}$ be three Galton-Watson processes indipendent with the same offspring distribution in formula (1). If $Z_0 = x + y$, $Z_0^{(1)} = x$ and $Z_0^{(2)} = y$, then $\forall\ n > 0$

$$Z_n \overset{d}{=} Z_n^{(1)} + Z_n^{(2)}. \tag{2}$$

Continuous-state Branching Process and Limit theorem

$$\{X(t)\}_{t\in\mathbb{R}^+}, \text{ with } X(t) \in \mathbb{R}^+.$$

Branching property

For each initial condition x, $y \in \mathbb{R}^+$, then for any fixed $t \geq 0$, the markov transition kernels satisfy

$$P_t(x+y,\cdot) = P_t(x,\cdot) * P_t(y,\cdot)$$

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Theorem (Aliev, Shurenkov (1982))

Let $\{X(t)\}_{t\geq 0}$ be a CSBP, then there exists a sequence of GW processes $\{Z^{(k)}(n)\}_{n\geq 0}$ for $k \geq 0$ such that

$$\left\{\frac{Z^{(k)}(\lfloor\gamma_k t\rfloor)}{k}\right\}_{t\geq 0} \Longrightarrow \{X(t)\}_{t\geq 0} \text{ for } k \to \infty, \tag{3}$$

where $\frac{Z^{(k)}(0)}{k} \longrightarrow X(0)$; γ_k is a sequence of positive reals tending to infinity and $\Longrightarrow$ means weak convergence in Skorhokod space $\mathbb{D}([0,\infty),\mathbb{R}^+)$.

How can we provide a GW process with memory?

We set random waiting times between successive generations of each Galton-Watson process $\{Z(n)\}_{n\in\mathbb{N}}$.

Let $J_1, J_2, \dots$ be i.i.d. random waiting times, then we define the processes

$$\{T(n)\}_{n\geq 0} \text{ t.c. } \begin{cases} T(0) = 0; \\ T(n) = \sum_{i=1}^{n} J_i. \end{cases} \tag{4}$$

$$N(t) = \sup\{n \geq 0 \, : \, T(n) \leq t\}. \tag{5}$$

The main idea is to consider the modified process

$$\{Z(N(t))\}_{t\in\mathbb{R}^+}.$$

Infinite mean waiting times

Let $J_1, J_2, \ldots$ be waiting times belonging to $DOA(D)$, with D stable r.v. with index $\beta \in (0,1)$. There exists a sequence $b(n)$ s.t.

$$b(n)T(n) \xrightarrow{d} D \tag{6}$$

and

$$\{b(n)T(\lfloor nt \rfloor)\}_{t\geq 0} \Longrightarrow \{D(t)\}_{t\geq 0}$$

where $D(t)$ is a *stable subordinator* with index β.

Theorem (Becker-Kern, Meerschaert, Scheffler (2004))

There exists a sequence $\tilde{b}(n)$ s.t.

$$\left\{ \frac{1}{\tilde{b}(n)} N(\lfloor nt \rfloor) \right\}_{t\geq 0} \Longrightarrow \{E(t)\}_{t\geq 0} \tag{7}$$

in $\mathbb{D}([0,\infty), \mathbb{R}^+)$, where $\{E(t)\}_{t\geq 0}$ is the inverse subordinator defined as

$$E(t) = \inf\{x,\ D(x) > t\}. \tag{8}$$

Limit in the product space

For the GW process we have

$$\left\{ \frac{Z^{(k)}(\lfloor \gamma_k t \rfloor)}{k} \right\}_{t\geq 0} \Longrightarrow \{X(t)\}_{t\geq 0} \text{ per } k \to \infty \tag{9}$$

and for the waiting times we have

$$\left\{ \frac{1}{\tilde{b}(k)} N(\lfloor kt \rfloor) \right\}_{t\geq 0} \Longrightarrow \{E(t)\}_{t\geq 0} \text{ per } k \to \infty. \tag{10}$$

Theorem

Let $\{Z^{(k)}(n)\}_{n\in\mathbb{N}}$ a sequence of GW s.t. limit (9) holds, let $J_1, J_2, \ldots$ i.i.d. waiting times belonging to the DOA of a stable law, such that limit (10) holds, then

$$\left(\frac{Z^{(k)}(\lfloor \tilde{b}_k t \rfloor)}{k}, \frac{N(\lfloor kt \rfloor)}{\tilde{b}_k} \right) \Longrightarrow (X(t), E(t)) \tag{11}$$

in the product space $\mathbb{D}([0,\infty), \mathbb{R}^+) \times \mathbb{D}([0,\infty), \mathbb{R}^+)$.

Composition map

Composition map

$$
\begin{aligned}
C: \quad \mathbb{D}([0,\infty),\mathbb{R}^+) \times \mathbb{D}([0,\infty),\mathbb{R}^+) &\to \mathbb{D}([0,\infty),\mathbb{R}^+) \\
(x,y) &\to x(y)
\end{aligned}
$$

is continuous when applied to this processes.

Theorem

Under previous hypothesis, the convergence of the rescaled compound process holds

$$\left\{ \frac{Z^{(k)}\left(N(\lfloor kt \rfloor)\right)}{k} \right\}_{t\geq 0} \Longrightarrow \{X\left(E(t)\right)\}_{t\geq 0}$$

for $k \to \infty$, in $\mathbb{D}([0,\infty),\mathbb{R}^+)$.

Branching property

The CSBP with memory $\{X(E(t))\}_{t\geq 0}$ does not satisfy the classical branching property, however it holds a modified version with the conditional expectation:

$$\mathbb{E}\left[\mathbb{E}_{x+y}\left[e^{-\lambda X(E(t))}\middle| E(t)\right]\right] = \mathbb{E}\left[\mathbb{E}_{x}\left[e^{-\lambda X(E(t))}\middle| E(t)\right]\mathbb{E}_{y}\left[e^{-\lambda X(E(t))}\middle| E(t)\right]\right],$$

for all $x, y > 0$ and all $\lambda > 0$

First and second moments of CSBP with memory

Theorem

The first and the second moments of the process $\{X(E(t))\}_{t\geq 0}$, when they exist, have explicit form

$$\mathbb{E}_x[X(E(t))] = xE_\beta(-bt^\beta),$$

$$\mathbb{E}_x\left[X(E(t))^2\right] = \begin{cases} x^2 + x\tilde{\beta}\frac{\Gamma(2)}{\Gamma(\beta+1)}t^\beta & \text{if } b = 0 \\ x^2E_\beta(-2bt^\beta) - \frac{\tilde{\beta}x}{b}\left(E_\beta(-2bt^\beta) - E_\beta(-bt^\beta)\right) & \text{if } b \neq 0 \end{cases}$$

where E_β is the Mittag-Leffler function of order β, $b > 0$ and $\tilde{\beta}$ are parameters of the CSBP, from the branching mechanism.

Remark: Mittag-Leffler function is

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta k + 1)}. \tag{12}$$

Particular case: Feller branching diffusion

Let $\{X_t\}_{t\geq 0}$ be the solution to SDE

$$dX_t = bX_t dt + \sqrt{\tilde{\beta} X_t} dW_t, \tag{13}$$

where $\{W_t\}_{t\geq 0}$ is a standard BM and the initial condition is $X_0 = x$. The transition density $p_x(y,t)$ satisfies the Fokker-Planck equation

$$\partial_t p_x(y,t) = -bp_x(y,t) + (\tilde{\beta} - by)\frac{\partial}{\partial y} p_x(y,t) + \frac{\tilde{\beta} y}{2}\frac{\partial^2}{\partial y^2} p_x(y,t).$$

Theorem

Under previous hypothesis, let $\{X(E(t))\}_{t\geq 0}$ be a CSBP with memory, with $\{X(t)\}_{t\geq 0}$ Feller branching diffusion. If there exist the density $m_x(y,t)$, then it satisfies the fractional differential equation:

$$\partial_t^\beta m_x(y,t) = -bm_x(y,t) + (\tilde{\beta} - by)\frac{\partial}{\partial y} m_x(y,t) + \frac{\tilde{\beta} y}{2}\frac{\partial^2}{\partial y^2} m_x(y,t),$$

with ∂_t^β is the Caputo fractional derivatives.

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Thank you!

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