Prof. Dr. M. Disertori
Raphael Winter
Institut für Angewandte Mathematik
Universität Bonn

PDE and Modelling

Sheet 1, due April 22

Problem 1 (Dimensional Analysis *2+2+2+4 points*)

Consider a one dimensional system described by a mass density $\varrho(t,x,v)$ on phase space. So $x \in \mathbb{R}$ denotes the position (e.g. of a particle) and $v \in \mathbb{R}$ the velocity. Further let $\phi(x)$ be a given potential with acceleration field $-\phi'$ (dimensionality force/mass). Determine which of the following equations can be physically meaningful (D diffusion coefficient, dimensionality $\frac{L^2}{T^3}$):

(i) $\varrho_t + v\varrho_x - \phi'\varrho_v = 0$

(ii) $\varrho_t + v\varrho_x - \phi'\varrho_x = 0$

(iii) $\varrho_t + v\varrho_x = D\varrho_{vv}$.

For the equations that are physically meaningful, derive the adimensional equations.

Problem 2 (Rigid displacements *10 points*)

Let $\Omega \subset \mathbb{R}^n$ open and convex, and let $u : \Omega \to \mathbb{R}^n$ be a C^2 vector field. Prove that the following are equivalent:

(i) u is an infinitesimal rigid displacement, i.e. ∇u is constant and skew symmetric.

(ii) For all $p, q \in \Omega$,

$$(q-p)(u(q)-u(p)) = 0,$$

(iii) $\nabla u(p)$ is skew symmetric at each $p \in \Omega$.

Hint: for iii) to i), consider $\partial_j u_i + \partial_i u_j$, differentiate with respect to x_k. Do the same for permutations of i, j, k, and combine the equations.

Problem 3 (Two body problem *5+5 points***)**

We would like to model the movement of a planet with mass m_P around a sun with mass m_S interacting solely by gravitational forces. Their positions shall be given by $x_P(t)$ and $x_S(t)$, and their orientated distance by

$$x(t) = x_P(t) - x_S(t).$$

(i) Derive the equations of motion of the planet and the sun. Calculate the kinetic energy E_{kin}, the potential energy E_{pot} and the angular momentum L of the two body system. Verify that the total energy $E = E_{kin} + E_{pot}$ and the angular momentum are conserved quantities.

(ii) Show that the movement of the planet around the sun is described by the equation

$$m_P x''(t) = -G m_P m \frac{x(t)}{|x(t)|^3},$$

where $G > 0$ is the gravitational constant and $m = m_P + m_S$ is the total mass.

(iii) Prove Kepler's second law of planetary motion: The distance vector $x(t)$ between the planet and the sun sweeps out equal areas $A_{\Delta t}$ during equal intervals Δt of time.

Hint: Use and show, that for $\Delta t = t_2 - t_1$, there holds:

$$A_{\Delta t} = \frac{1}{2} \int_{t_1}^{t_2} |x(t) \times x'(t)| dt.$$

Problem 4 (Modelling* *6+4 extra points***)**

Assume you add a drop of water soluble dye to a large bowl of water and start rapidly heating it up on a stove. The goal is to predict how the dye diffuses, so the concentration of dye at different positions in the bowl at later times. Your model needs only to be valid for temperatures much lower than boiling temperature.

(i) Formulate the situation above as a PDE problem.

(ii) Discuss the validity of the approximations you made in the modelling process.

Hint: Start by modelling the heating process first (heat equation!). Only then model the diffusion of the dye. A change of temperature changes a coefficient in the equation for the diffusion of the dye, which one? The temperature on the surface of the bowl you can assume to be given. You can further assume that the bowl is some given precompact domain $\Omega \subset \mathbb{R}^3$.