Partial Differential Equations and Functional Analysis

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Problem Sheet 0

Problem 1. (compactness)

(i) Let $(X_1, \mathcal{T}_1), (X_2, \mathcal{T}_2)$ be topological spaces. Assume that $f : (X_1, \mathcal{T}_1) \to (X_2, \mathcal{T}_2)$ is continuous and $K \subset X_1$ is compact. Then $f(K) \subset X_2$ is compact.

(ii) Let $(X, \mathcal{T})$ be a topological space. Let $f : (X, \mathcal{T}) \to \mathbb{R}$ be continuous and $K \subset X$ be compact. Then f attains its maximum and minimum on K, i.e. there exist $a, b \in K$ such that

$$f(a) \leq f(x) \leq f(b) \quad \forall x \in K.$$

Problem 2. (injectivity and surjectivity of linear maps in infinite dimensional spaces)

Consider the space $l_\infty(\mathbb{R}) = \{f : \mathbb{N} \to \mathbb{R} : \sup_{i\in\mathbb{N}} |f_i| < \infty\}$ of bounded sequences.

(i) Find a linear map $L_1 : l_\infty(\mathbb{R}) \to l_\infty(\mathbb{R})$ that is injective but not surjective.

(ii) Find a linear map $L_2 : l_\infty(\mathbb{R}) \to l_\infty(\mathbb{R})$ that is surjective but not injective.

Problem 3. (Hölder and Minkowski inequality on l_p spaces)

For a sequence $f : \mathbb{N} \to \mathbb{R}$ and $1 \leq p < \infty$ we define

$$\|f\|_p = \left(\sum_{i\in\mathbb{N}} |f_i|^p \right)^{\frac{1}{p}}.$$

We define the space $l_p(\mathbb{R}) = \left\{ f : \mathbb{N} \to \mathbb{R} : \|f\|_p < \infty \right\}$.

(i) Let $1 < p < \infty$ and $q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Let $f \in l_p(\mathbb{R})$ and $g \in l_q(\mathbb{R})$. Then

$$\left| \sum_{i\in\mathbb{N}} f_i g_i \right| \leq \|f\|_p \|g\|_q .$$

Hint: It suffices to prove this for $\|f\|_p = \|g\|_q = 1$. Why? Also remember Young's inequality: $ab \leq \frac{1}{p}a^p + \frac{1}{q}b^q$.

(ii) Let $f, g \in l_p(\mathbb{R})$. Then

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p .$$

Hint: It is useful to write $|f_i + g_i|^p = |f_i + g_i| \, |f_i + g_i|^{p-1}$.

Problem 4. (Generating new metrics)

Consider a concave nondecreasing function $\psi \in C^2([0, \infty))$ with $\psi(0) = 0$ and $\psi(x) > 0$ for $x > 0$. Show that for any metric d the function $\psi \circ d$ is a metric, too. What happens if we replace the condition $\psi \in C^2([0, \infty))$ by $\psi \in C^0([0, \infty))$?