Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 6.

Due in class, Friday, November 24, 2017.

Problem 1. (1+2+2 points)

State whether or not there is a solution to the minimization problem

$$\min \left\{ \sup_{x\in[0,1]} |g(x) - f(x)| : f \in M_i \right\} \ , \ i = 1, 2, 3$$

for an arbitrary $g \in C([0,1])$, and prove your statement. The sets M_i are given by

(i) $M_1 := \left\{ f \in C([0,1]) \mid \sup_{x\in[0,1]} |f(x)| \le 1 \right\}$

(ii) $M_2 := \left\{ f \in C([0,1]) \mid \int_0^1 f(x)x^2 dx = 0 \right\}$
Hint: Set $d := \int_0^1 g(x)x^2 dx$. Note that $d = \int_0^1 (g-f)(x)x^2 dx$ for $f \in M_2$, and use this to derive first a lower bound for $\inf_{f\in M_2} \|f-g\|_{L^\infty([0,1])}$.

(iii) $M_3 := \left\{ f \in C([0,1]) \mid \int_0^1 f(x)(x - \frac{1}{2})dx = 0 \right\}$.
Hint: similar idea as for (ii).

Problem 2. (2+2+1 points)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex and continuous.

(i) Show that $f = \sup\{g(x) \mid g : \mathbb{R}^n \to \mathbb{R} \text{ affine}, g \le f\}$.
Hint: Show first that the epigraph $\{(x, y) \in \mathbb{R}^n \times \mathbb{R} \mid y \ge f(x)\}$ is closed and convex.

(ii) Show that the **subdifferential** $\partial^- f(x) = \{a \in \mathbb{R}^n \mid f(y) \ge f(x) + a \cdot (y - x) \text{ for all } y \in \mathbb{R}^N\}$ is nonempty for all $x \in \mathbb{R}^n$.
Hint: You can assume $x = 0$ (why?). Consider a sequence of affine functions g_k such that $g_k(x) \to f(x)$ from (i) and show compactness of the coefficients.

(iii) Let $(\Omega, \mathcal{A}, \mu)$ be a probability space, i.e. a measure space with $\mu(\Omega) = 1$, $X : \Omega \to \mathbb{R}^n$ a Borel measurable map with $\int_\Omega |X| \, d\mu < \infty$. Show that

$$f\left(\int_\Omega X \, d\mu \right) \le \int_\Omega f(X) \, d\mu.$$

This is **Jensen's inequality**.

Problem 3. (2+3 points)

(i) Let $T : C([0,1]) \to C([0,1])$ be given by

$$Tf(x) = \int_0^x f(y)\mathrm{d}y.$$

Show that $T(\overline{B}(0,1))$ is precompact.

(ii) Let $X, Y \subset \mathbb{R}^n$ be compact. Let $K \in C(X \times Y; \mathbb{R})$. Define $T : C(Y) \to C(X)$ by

$$T(f)(x) = \int_Y K(x,y)f(y)\mathrm{d}y \quad \text{for } f \in C(Y) \text{ and } x \in X.$$

Consider the closed unit ball $\overline{B}(0,1) \subset C(Y)$. Prove that $T(\overline{B}(0,1)) \subset C(X)$ is precompact in $C(X)$.
Hint: Arzela-Ascoli. Note that K is uniformly continuous on $X \times Y$.

Problem 4. (3+2 points)

(i) For $c > 0$ define

$$M_c := \left\{ f \in C^1([0,1]) \ : \ \int_0^1 |f(x)|^2 dx + \int_0^1 |f'(x)|^2 dx \leq c \right\}.$$

Prove that $\overline{M_c}$ is compact in $C([0,1])$.
Hint: Arzela-Ascoli.

(ii) Suppose E is a closed linear subspace of $C^1([0,1])$ such that there is $C > 0$ with

$$\int_0^1 |f'(x)|^2 dx \leq C \int_0^1 |f(x)|^2 dx \quad \text{for all } f \in E.$$

Prove that E is finite dimensional.
Hint: Use (i) to show that the closure of the unit ball in the C^0-norm is compact.