Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 4.

Due in class, Friday, November 10, 2017.

Problem 1. (1+1+1+2 points)

Let $f_1 = \chi_{(-\frac{1}{2},\frac{1}{2})}$. We define $f_k : \mathbb{R} \to \mathbb{R}$ for $k \geq 2$ recursively by $f_{k+1} = f_1 * f_k$.

(i) Compute f_2 and f_3.

(ii) Show that $g \in C_c(\mathbb{R})$ implies $f_1 * g \in C_c^1(\mathbb{R})$ and determine $(f_1 * g)'$.

(iii) Show $f_k \in C_c^{k-2}$ for $k \geq 2$.

(iv) Compute $\int_{\mathbb{R}} x f_k(x)\mathrm{d}x$ and $\int_{\mathbb{R}} x^2 f_k(x)\mathrm{d}x$ for $k = 1, 2, 3$ (if you want you can first derive a general formula for $\int_{\mathbb{R}} x(f * g)(x)\mathrm{d}x$ and $\int_{\mathbb{R}} x^2(f * g)(x)\mathrm{d}x$ by using the change of variables $x = x' + y', y = y'$).

Problem 2. (5 points)

For which $\alpha > 0$ and $p \in [1, \infty)$ is

$$f(x) = \frac{1}{|x|^\alpha}$$

in $W^{1,p}(B(0,1))$?
Hint: Consider first the expression $\int_{B(0,1)\setminus B(0,\varepsilon)} f \partial_i \varphi$ *for* $\varphi \in C_c^\infty(B(0,1))$ *and let* $\varepsilon \searrow 0$.

Problem 3. (5 points)

Let $L > 0$, $p \in [1, \infty)$ and suppose that $U \subset \mathbb{R}^n$ is open, and $U \subset \mathbb{R}^{n-1} \times (0, L)$. Prove that for all $u \in C_c^\infty(U) \cap W^{1,p}(U)$

$$\int_U |u(x)|^p \,\mathrm{d}x \leq \frac{L^p}{p} \int_U |\nabla u(x)|^p \,\mathrm{d}x.$$

Hint: Show first that for q with $\frac{1}{p} + \frac{1}{q} = 1$, for every $x = (x', x_n) \in U$, $x' \in \mathbb{R}^{n-1}$,

$$\left|u(x', x_n)\right|^p \leq x_n^{p/q} \int_0^L \left|\frac{\partial}{\partial x_n} u(x', t)\right|^p \mathrm{d}t.$$

Problem 4. (1+3+1 points)

Let $g \in L^1((0,1))$. Define $f(x) = \int_0^x g(t)\mathrm{d}t$.

(i) Show that $f \in W^{1,1}((0,1))$ with weak derivative $f' = g$.
Hint: Start from $\int_0^1 f(x)\varphi'(x)\mathrm{d}x$ *and use Fubini.*

(ii) Let $h \in W^{1,1}((0,1))$ with weak derivative $h' \in L^1((0,1))$. Show that there is $c \in \mathbb{R}$, s.t. $h(x) = \int_0^x h'(t)\,dt + c$ almost everywhere. In particular h has a continuous representative.
Hint: Show first that $h' = 0$ *implies that* h *is constant a.e. .*

(iii) Let $h \in W^{1,p}((0,1))$ with $p \in (1,\infty]$. Show that h has a representative in $C^{0,1-1/p}([0,1])$.
Hint: Use Hölder's inequality.