Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 2.

Due in class, Friday, October 27, 2017.

Problem 1. (2+3+5∗ points)

Let (X, d) be a metric space. We define the Hausdorff distance of two sets $A \subset X$ and $B \subset X$ by

$$d_H(A, B) := \inf \{r > 0 : A \subset B_r(B) \text{ and } B \subset B_r(A)\}.$$

(i) Show that

$$d_H(A, B) = \max \left(\sup_{x \in A} \operatorname{dist}(x, B), \sup_{y \in B} \operatorname{dist}(y, A) \right).$$

Let $\mathcal{C} := \{A \subset X : A \text{ closed, bounded, non-empty}\}$.

(ii) Show that d_H is a metric on $\mathcal{C}$.

(iii∗) Show that $(\mathcal{C}, d_H)$ is complete if (X, d) is complete.
Hint: see lecture.

Problem 2. (5 points)

Let $(X, \|\cdot\|)$ be a normed space. Prove that $(X, \|\cdot\|)$ is a Banach space if and only if every absolutely convergent series converges, i.e. for $x : \mathbb{N} \to X$ we have

$$\sum_{k=0}^{\infty} \|x_k\| < \infty \quad \Rightarrow \quad \sum_{k=0}^{\infty} x_k \quad \text{converges in } X.$$

Hint: To prove completeness first show that a Cauchy sequence has a subsequence with $\left\|x_{n_{k+1}} - x_{n_k}\right\| \leq 2^{-k}$.

Problem 3. (1+1+3 points)

Define $d : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ by

$$d(x, y) = \sum_{i=1}^{\infty} 2^{-i} \frac{|x_i - y_i|}{1 + |x_i - y_i|} .$$

(i) Prove: d is a metric.

(ii) Let $\{x^{(j)}\}_{j=1}^{\infty} \subset \mathbb{R}^{\mathbb{N}}$ be a sequence in $\mathbb{R}^{\mathbb{N}}$. Prove that $d(x^{(j)}, 0) \to 0$ if and only if $x_k^{(j)} \to 0$ for all $k \in \mathbb{N}$.

(iii) Prove: There is no norm on $\mathbb{R}^{\mathbb{N}}$ which induces the same topology as d.
Hint: Let $e_j = (0, \ldots, 1, \ldots) \in \mathbb{R}^{\mathbb{N}}$ *be the* j*-th unit vector. Let* $\|\cdot\|$ *be some norm. Construct a sequence* $x^{(j)} = a_j e_j, a_j \in \mathbb{R}$, *s.t.* $\limsup_{j \to \infty} \left\|x^{(j)}\right\| \neq 0$.

Problem 4. (5 points)

Let $(H, (\cdot, \cdot))$ be a real Pre-Hilbert space, and let $T : H \to H$ be an isometry, i.e. $\|T(x) - T(y)\| = \|x - y\|$ for all x, $y \in H$. Prove that T is affine, i.e. there is a linear map $L : H \to H$ and $x_0 \in H$ such that $T(x) = Lx + x_0$ for all $x \in H$.

Hint: If $T(0) = 0$ use the identity $\|T(x) - T(y)\|^2 = \|x - y\|^2 = \|x\|^2 - 2\,(x, y) + \|y\|^2$ to show that $(T(x), T(y)) = (x, y)$ for all x, $y \in H$. Then compute $\|T(x + y) - T(x) - T(y)\|^2$ etc.