Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 13.

Due in class, Friday, January 26, 2018.

The points on this sheet are the last ones relevant for the admission to take the exam.

Problem 1. (2+2+1 points)

Let $U \subset \mathbb{R}^n$ be open and bounded and assume that $f \in L^2(U)$. For $u \in W_0^{1,2}(U)$ let

$$E(u) := \frac{1}{2}\int_U |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n - \int_U fu \mathrm{d}\mathcal{L}^n$$

Let $M \subset W_0^{1,2}(U)$ be closed, convex and nonempty. Prove the following assertions:

(i) The functional E attains its minimum in M, i.e., there exists $u \in M$ such that

$$E(u) \leq E(v) \quad \forall v \in M. \tag{1}$$

Hint: Use the so-called direct method. Take a sequence $v_k \in M$ such that $\lim_{k\to\infty} E(v_k) = \inf_{v\in M} E(v)$. Prove that the sequence is bounded in $W_0^{1,2}(U)$ and extract a weakly convergent subsequence. Then show that the limit is in M and satisfies (1).

(ii) An element $u \in M$ is a minimizer of E if and only if u satisfies the variational inequality

$$\int_U \sum_{i=1}^n \partial_i(u-v)\partial_i u - (u-v)f \mathrm{d}\mathcal{L}^n \leq 0 \quad \forall v \in M. \tag{2}$$

Hint: Set $v_t = u - t(u-v)$ and show that $\frac{1}{t}(E(u) - E(u_t)) \leq 0$. Consider $t \to 0$.

(iii) If M is a closed subspace then (2) is equivalent to the weak form of the Euler-Lagrange equation

$$\int_U \sum_{i=1}^n \partial_i w \partial_i u - wf \mathrm{d}\mathcal{L}^n = 0 \quad \forall w \in M.$$

Problem 2. (3+2 points)

Let $U = B(0,1) \subset \mathbb{R}^n$ and $1 \leq p < n$.

(i) Define $u_k : U \to \mathbb{R}$ by

$$u_k(x) := \begin{cases} k^{\frac{n-p}{p}}(1-k|x|) & \text{if } |x| < \frac{1}{k} \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\{u_k : k \in \mathbb{N}\}$ is bounded in $W^{1,p}(U)$ but does not admit a subsequence that converges strongly in $L^{p^*}(U)$, where $p^* = \frac{np}{n-p}$.

(ii) Let $u : U \to \mathbb{R}$ be given by $u(x) := \log(\log(1 + \frac{1}{|x|}))$ if $x \in U \setminus \{0\}$, and $u(0) = 0$. Prove that $u \in W^{1,n}(U)$ but $u \notin L^\infty(U)$.

Hint: You do not need to check that the pointwise derivative is indeed the weak one.

Problem 3. (1+2+2 points)

Suppose H is a Hilbert space with orthonormal basis $\{e_i : i \in \mathbb{N}\}$, and let $T \in \mathcal{L}(H)$ be a *Hilbert-Schmidt operator*, i.e. such that $\sum_{i=1}^{\infty} \|Te_i\|^2 < \infty$.

(i) Prove that for every orthonormal basis $\{f_i : i \in \mathbb{N}\}$ of H

$$\|T\|_{HS} := \left(\sum_{i=1}^{\infty} \|Te_i\|^2\right)^{1/2} = \left(\sum_{i=1}^{\infty} \|Tf_i\|^2\right)^{1/2}, \quad \text{and} \quad \|T\| \leq \|T\|_{HS}.$$

(ii) Prove that T is compact.

Hint: Use that T can be approximated by operators of finite dimensional range. How?

(iii) Let $H := L^2([0,1])$, and $K \in L^2([0,1]^2)$. Define the integral operator $T \in \mathcal{L}(L^2([0,1]))$ by

$$Tf(x) := \int_0^1 K(x,t) f(t)\, dt.$$

Prove that T is a Hilbert-Schmidt operator and $\|T\|_{HS} = \|K\|_{L^2}$.

Hint: For given $x \in [0,1]$ consider $K(x,\cdot) \in L^2([0,1])$ and write down the representation formula.

Problem 4. (5 points)

Let $1 < p < \infty$. We denote the terms of a sequence $x : \mathbb{N} \to l_p$ by $x^{(1)}, x^{(2)}, \ldots$. Prove that for a sequence $x : \mathbb{N} \to l_p$ and $x^* \in l_p$

$$x^{(n)} \rightharpoonup x^* \text{ in } l_p \quad \Longleftrightarrow \quad x_j^* = \lim_{n\to\infty} x_j^{(n)} \text{ for all } j \in \mathbb{N} \text{ and } \sup_{n\in\mathbb{N}} \|x^{(n)}\|_{l_p} < \infty.$$

Hint: For the backward implication use Lemma 8.16.