Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 10.

Due in class, Friday, December 22, 2017.

Problem 1. (3+2 points)

(i) Let X be a real vector space, and let $p : X \to \mathbb{R}$ be sublinear. Suppose $S \subset X$ is a subspace, and let $f : S \to \mathbb{R}$ be linear, and such that $f(s) \leq p(s)$ for all $s \in S$. Let G be an Abelian semigroup of linear operators on X with identity element, i.e., $G \subset \mathcal{L}(X)$, A, $B \in G \Rightarrow AB = BA \in G$, and $id \in G$. Suppose that for all $A \in G$ we have $p(Ax) \leq p(x)$ for all $x \in X$, and $s \in S \Rightarrow As \in S$ and $f(As) = f(s)$.
Prove that there exists $F : X \to \mathbb{R}$ linear such that

(1) $F(s) = f(s)$ for all $s \in S$,

(2) $F(x) \leq p(x)$ for all $x \in X$, and

(3) $F(Ax) = F(x)$ for all $x \in X$ and all $A \in G$.

Hint: Set $q(x) := \inf\{\frac{1}{n}(p(A_1x + \cdots + A_nx)) \ : \ n \in \mathbb{N}, A_i \in G\}$. *Show that q is sublinear, and that $q \leq p$. Use Hahn-Banach to obtain a suitable extension F. To see that $F(Ax) = F(x)$, observe that* $F(x) - F(Ax) \leq q(x - Ax) \leq \frac{1}{n}p(x - A^{n+1}x) \to 0$ *as* $n \to \infty$.

(ii) Show that there exists a finitely additive translation-invariant measure μ defined on all bounded subsets of $\mathbb{R}$ such that $\mu(E)$ agrees with the Lebesgue measure for every bounded Lebesgue measurable subset E.

Hint: Let X be the space of compactly supported bounded functions on $\mathbb{R}$, and let Y be the subset of Lebesgue measurable, compactly supported, bounded functions. Define $T(f) := \int_{\mathbb{R}} f \, d\mathcal{L}$ *on Y, and use (i).*

Problem 2. (5 points)

Let $\mathcal{M}_n$ denote the Lebesgue measurable subsets of $\mathbb{R}^n$. For $E \in \mathcal{M}_n$ let
$\mathcal{M}_n(E) = \{A \in \mathcal{M}_n : A \subset E\}$ and

$$\begin{aligned} \mathrm{ba}\,(E, \mathcal{L}^n) \quad := \quad & \{\lambda : \mathcal{M}_n(E) \to \mathbb{R} : \lambda \text{ finitely additive, } \|\lambda\|_{\mathrm{var}}(E) < \infty, \\ & \mathcal{L}^n(N) = 0 \implies \lambda(N) = 0\}. \end{aligned}$$

Then for $f \in L^\infty(E)$ define the integral $\int_E f \, d\lambda$ as usual by first considering finite linear combinations of characteristic functions. The condition $\mathcal{L}^n(N) = 0 \implies \lambda(N) = 0$ guarantees that this integral depends only on the equivalence class of of f where $f \sim g$ if $f = g$ $\mathcal{L}^n$ a.e. Show that the map $J : \mathrm{ba}\,(E, \mathcal{L}^n) \to (L^\infty(E))'$ given by

$$J(\lambda)(f) = \int_E f \, d\lambda$$

is a linear isometry.

Problem 3. (5 points)

Let X be a Banach space, and suppose $T \in \mathcal{L}(X)$ is such that for every $x \in X$ there exists $n \in \mathbb{N}$ with $T^n x = 0$. Prove that there exists $N \in \mathbb{N}$ with $T^N = 0$.

Hint: Note that $X = \bigcup_{n\in\mathbb{N}} \mathcal{N}(T^n)$, and use Baire's theorem to find a candidate for N.

Problem 4. (3+1+1 points)

(i) For $n \in \mathbb{N}$ set

$$M_n := \{f \in C([0,1]) \;:\; \exists\, 0 \le x^* \le 1 - \frac{1}{n} \text{ s.t. } |f(x^* + h) - f(x^*)| \le nh \text{ for all } 0 < h < 1 - x^*\}.$$

Prove that M_n is closed and nowhere dense in $C([0,1])$

Hint: (Closed) If $f_k \to f$, extract a subsequence with $x^*_{k_\ell} \to x^* \in [0, 1 - \frac{1}{n}]$, and observe that $0 < h < 1 - x^*$ implies $0 < h < 1 - x^*_{k_\ell}$ for large ℓ. (Nowhere dense) For $f \in M_n$ and $\epsilon > 0$ construct a continuous function g with $\|f - g\|_\infty < \epsilon$ and $|g'_+| > n$.

(ii) Consider

$$M := \{f \in C([0,1]) \;:\; \text{there exists } x^* \in [0,1) \text{ such that the right derivative } f'_+(x^*) \text{ exists}\}.$$

Prove that M is meagre in $C[0,1]$.

Hint: Show first that $M \subset \bigcup_{n\in\mathbb{N}} M_n$.

(iii) Prove that there exists a function $f \in C([0,1])$ that is nowhere differentiable.