Partial Differential Equations and Functional Analysis

Winter 2017/18
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Problem Sheet 1.

Due in class, Friday, October 20, 2017.

Problem 1. (2+3+2 points)

We define

$$\mathcal{T} := \{ U \subset \mathbb{R} \ : \ U = \emptyset \text{ or } \mathbb{R} \setminus U \text{ is countable } \}.$$

Here we call a set A countable if A is empty or finite or if there is a bijective map $j : \mathbb{N} \to A$.

(i) Prove that $\mathcal{T}$ is a topology on $\mathbb{R}$.

(ii) Prove that a sequence $x : \mathbb{N} \to \mathbb{R}$ converges to $x^\star \in \mathbb{R}$ with respect to $\mathcal{T}$ if and only if there is $k_0 \in \mathbb{N}$ such that $x_k = x^\star$ for all $k \geq k_0$.
Hint: To prove "only if" consider first sequences for which $x_k \neq x^\star$ for all $k \in \mathbb{N}$.

(iii) Show that there exists $A \subset \mathbb{R}$ which is (with respect to $\mathcal{T}$) sequentially closed but not closed.
You may use the result from (ii) even if you did not prove it.

Problem 2. (2+2 points)

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, and let $\mathcal{T}$ be the product topology on $X \times Y$ as defined in class (see Example 1.2 (vii)).

(i) Prove that the product topology $\mathcal{T}$ is the coarsest topology for which the canonical projections $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ are continuous, where $p_X(x, y) := x$ and $p_Y(x, y) := y$ for all $(x, y) \in X \times Y$.

(ii) Prove that $W \in \mathcal{T}$ if

$$\forall (x, y) \in W \quad \exists\, U \in \mathcal{T}_X,\ V \in \mathcal{T}_Y \ : \ (x, y) \in U \times V \subset W \ .$$

Problem 3. (2+2 points)

(i) Prove that any interval $[a, b]$, $a < b$, is connected in $\mathbb{R}$ (with the standard topology).
Hint: Assume that $[a, b] = U \cup V$ where U and V are open and closed (with respect to the relative topology). Pick $u \in U$ and $v \in V$ (wlog $u < v$) and consider $w := \sup([u, v] \cap U)$.

(ii) A topological space $(X, \mathcal{T}_X)$ is called path connected if and only if for all x, $y \in X$ there exists a continuous function from $[0, 1]$ (with the standard topology) to X such that $f(0) = x$ and $f(1) = y$. Prove that every path connected space is connected.
You may use the result from (i) even if you did not prove it.

Problem 4. (2+3 points)

(i) Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and $n \in \mathbb{N}$. We set $d : \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{R}$,

$$d(x, y) := \#\{i \ : \ x_i \neq y_i\} \quad , \quad \text{where } x = (x_1, \ldots, x_n) \text{ and } y = (y_1, \ldots, y_n) \ .$$

Here, $\#M$ denotes the number of elements of a finite set M. Prove that d is a metric on $\mathbb{K}^n$.

(ii) Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d(x, y) := |\arctan(x) - \arctan(y)|$. Prove that d is a metric on $\mathbb{R}$, and that the induced topology agrees with the standard topology on $\mathbb{R}$. Show further that $(\mathbb{R}, d)$ is not complete, i.e. there is a sequence $x : \mathbb{N} \to \mathbb{R}$ which is Cauchy ($\forall \epsilon > 0\ \exists\ N \in \mathbb{N}\ :\ n, m > N \Rightarrow d(x_n, x_m) < \epsilon$) but which does not converge with respect to d.

Hint: You may use that (with respect to the standard topology) $\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ *is a homeomorphism, and* $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$.