Topics in functional integration and multiscale analysis

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Chapter 1

Introduction

The main subject of this course is the study of certain functional integrals arising in statistical mechanics and physics.

1.1 Some motivation: an example of Gibbs measure

Let us consider the atoms in a perfect cristal. At equilibrium the atoms are located at the sites $x \in \Lambda$ where $\Lambda \subset \mathbb{Z}^d$ is a finite set. Thermal fluctuations and other perturbations may cause the atoms to move a bit away from their equilibrium position. The atom at site x is displaced to a postition $x + \vec{\varphi}(x)$, where $\vec{\varphi}(x) \in \mathbb{R}^d$. The collection of all displacements $\{\varphi(x)\}_{x\in\Lambda}$ is called a *field* (vector valued)

$$\begin{aligned} \varphi: \quad & \Lambda \to \mathbb{R}^d \\ & x \to \varphi(x) \end{aligned}$$

Each function $\varphi \in (\mathbb{R}^d)^\Lambda$ is a possible configuration for the deformed cristal. The set of all possible configurations will be denoted by

$$\Omega = (\mathbb{R}^d)^\Lambda.$$

Atoms "prefer" to remain near their equilibrium position so it takes some effort to displace them. This is encoded in the energy functional

$$\begin{aligned} H_\Lambda: \quad & (\mathbb{R}^d)^\Lambda \to \mathbb{R} \\ & \varphi \to H(\varphi) = \tfrac{1}{2} \textstyle\sum_{x\sim y\in\Lambda} \|\vec{\varphi}(x) - \vec{\varphi}(y)\|^2 \end{aligned} \tag{1.1.1}$$

where we use the L_2 norm $\|v\|^2 = \sum_{j=1}^d v_j^2$ and $x \sim y$ denotes a pair of nearest neighbors on the lattice $\|x - y\| = 1$. Note that if we deform each atom by the same amount $\vec{\varphi}(x) = \vec{\varphi}\ \forall x$ then we are doing a global translation of the cristal (no deformation) and the corresponding energy is zero.

We assign to each configuration φ a weight (probability density) proportional to $exp[-\beta H_\Lambda(\varphi)]$ where $\beta = 1/T$ and T is the temperature. Is this choice consistent with our intuition? For a large deformation the energy is large and the corresponding probability is small, as we should expect since it is "hard" to deform a cristal. The insertion of the β parameter is also consistent. Indeed for small temperature the atoms are "frozen" and moving them is "hard": the corresponding β is large thus giving a small probability. For high temperature the atoms are "exited" and move very easily: the corresponding β is small thus giving a large probability.

Boundary conditions The cristal is connected to the external world through the boundary of the volume Λ. This interaction translates into *boundary conditions* on H_Λ. The corresponding energy functional will be denoted by $H_\Lambda^{(bc)}$.

All the above arguments can be made precise by introducing a probability measure on $(\Omega, \mathcal{F})$ defined by

$$d\mu_{\Lambda,\beta}^{(bc)}(\varphi) = \frac{e^{-\beta H_\Lambda^{(bc)}(\varphi)}}{Z_{\Lambda,\beta}} d\varphi \tag{1.1.2}$$

where Ω is the set of all possible configurations (in our case deformations of the cristal), $\mathcal{F}$ is a $\sigma-$algebra on Ω and $d\varphi = \prod_{x\in\Lambda} \prod_{j=1}^d d\varphi_j(x)$ is the Lebesgue measure. Finally $Z_{\Lambda,\beta}$ is the normalization constant ensuring that $\mu_{\Lambda,\beta}^{(bc)}(\Omega) = 1$. This constant is called the *partition function*

$$Z_{\Lambda,\beta} = \int_\Omega e^{-\beta H_\Lambda^{(bc)}(\varphi)} d\varphi.$$

Remark 1 If we insert in the definition above the energy functional (1.1.1), the corresponding integral is divergent! The boundary conditions will ensure the integral is finite.

Remark 2 The energy functional (1.1.1), is a quadratic form

$$H_\Lambda(\varphi) = (\varphi, A\varphi) = \sum_{x,y\in\Lambda} \varphi(x) A_{xy} \varphi(y)$$

where for any pair x, y of sites not on the boundary of Λ we have

$$A_{xy} = \begin{cases} -1 & \|x-y\| = 1 \\ 2d & x = y \\ 0 & \|x-y\| > 1 \end{cases}$$

The corresponding measure (1.1.2) is called a *gaussian measure*. Most of the problems we will consider will be given by some form of gaussian measures, or perturbations of gaussian meausures.

1.2 Thermodynamic limit

The measure defined in (1.1.2) is called a *finite volume Gibbs measure*. If now we take a sequence of growing volumes Λ_n with $\Lambda_n \to \mathbb{Z}^d$ we can ask the following quastions

- Does the sequence of measures converge to some infinite volume measure?
- If yes, does the limit depend on the choice of the boundary conditions?

The answer to this question gives informations on the existence of a phase transition in the model (ex: liquid/gas or cristal/liquid).

1.3 Functional integrals

The object of this course if the study of a class of *functional integrals* of the type (1.1.2) . These are integrals over spaces of *functions*.

Going back to our initial example, cristal deformations, let us consider $d = 1$ and $\Lambda = (1, 2, \ldots, N)$. An element of the space of configurations $\varphi \in \Omega$ is a function *function* $\varphi : \Lambda \to \mathbb{R}$, but we can see it also a set on N real numbers $\varphi = (\varphi(1), \ldots, \varphi(N))$ corresponding to the values of the function at each point. Any function

$$F: \quad \begin{array}{l} \Omega \to \mathbb{R} \\ \varphi \to F(\varphi) \end{array}$$

can be seen as a function on N real variables $F(\varphi) = F(\varphi(1), \ldots, \varphi(N))$. The Lebesgue measure on Ω is then the Lebesgue measure on the product space $\mathbb{R}^N$.

$$d\varphi = \prod_{j=1}^{N} d\varphi(j).$$

Using Fubini's theorem we can define the integral

$$\int f(x_1, \ldots, x_N) dx_1 \cdots dx_N$$

indipendently of the integration order, for any integrable function $f : \mathbb{R}^N \to \mathbb{R}$ ($\int |f|$ is finite). This construction can be generalized to a *countable* set of variables.

When $d > 1$, let Λ a *finite* set of sites in $\mathbb{Z}^d$. At each site we have d variables $\varphi_1(x), \ldots .\varphi_d(x)$. Since Λ is finite we can define the product Lebesgue measure on the $d|\Lambda|$ variables corresponding to a cristal deformation φ. The thermodynamic limit then can be seen as the problem of defining the integral over an infinite number of variables.

Functions in the continuum . The arguments above concern only spaces of discrete functions $\varphi = \{\varphi(x)\}_{x\in\Lambda}$, where Λ is a set of lattice sites. Let us suppose now that the atoms are not in a solid phase but rather in a gas state. Then each atom could be anywhere in a region $\Lambda \subset \mathbb{R}^d$ of finite volume. The function $\varphi(x) \in \mathbb{R}^+$ may represent the number of particles in a small neighborhood of x. Then a configuration of the system is given by $\varphi = \{\varphi(x)\}_{x\in\Lambda}$ an *uncountable* set of real variables. Measure theory teaches us how to construct sigma-algebras out of countable products. In order to make sense of a measure on a uncountable product space we introduce spaces of distributions.

1.4 Role of the boundary conditions

Boundary conditions fix how the region Λ we are studying is connected to the outside. In our example we could say that on the boundary of our cristal we are attached to a very stable material where atoms are practically frozen. Then we have $\varphi(x) = 0$ for all sites x on the boundary of Λ (Dirichlet type b.c.)

The role of b.c. in the measure $d\mu_\Lambda^{(b.c)}$ when Λ gets big is analog to the initial conditions in a PDE. let us consider two famous PDEs: the heat and wave equations in one dimension:

$$\begin{aligned} \partial_t u(x,t) &= \alpha^2 \partial_x^2 u(x,t), \qquad u(x,0) = u_0(x), \quad x \in \mathbb{R}, t \geqslant 0, \\ \partial_{tt} u(x,t) &= v^2 \partial_x^2 u(x,t), \qquad u(x,0) = \phi(x),\ u_t(x,t) = \psi(x) \quad x \in \mathbb{R}, t \geqslant 0. \end{aligned}$$

The solution for the first (heat) equation is independent of the details of the initial condition $u_0(x)$: in particular any irregularities of u_0 are instantaneously smoothed out. This means we loose information. On the contrary, the solution of the second (wave) equation depends very strongly from the initial conditions. Actually, in this case the initial profile ϕ travels without ever changing shape. This means information is transferred without losses.

In the language of measures, the independence of the limit from the boundary conditions means there is only one possible measure describing our system at very large volume (one possible phase) , the dependence means that there are several possible measures at large volume (hence several possible phases).

1.5 Multiscale analysis

Let us look again at the cristal energy $H_\Lambda^{(b.c)}(\varphi) = \frac{1}{2}\sum_{x\sim y\in\Lambda} \|\vec{\varphi}(x) - \vec{\varphi}(y)\|^2$ with $\varphi(x) = 0$ for all x on the boundary, meaning that $x \in \Lambda$ but there exists at least one site $y \in \Lambda^c$ with $x \sim y$. In this functions only nearest neighbor sites $x \sim y$ interact. Then the corresponding density $e^{-\beta H_\Lambda^{(bc)}}$ is maximal when the variables φ are approximately constant on small regions (otherwise the probability is small). In other words the integral over $d\mu$ is concentrated around regions in Ω corresponding to configurations that are "locally constant".

In particular the boundary conditions can affect only a small number of sites near to the boundary of Λ. If the fraction of sites on the boundary $|\partial\Lambda|/|\Lambda|$ vanishes as $\Lambda \to \mathbb{Z}^d$, (take for instance a growing sequence of cubes), then it is reasonable to expect that boundary conditions will have no influence on the limit. Boundary conditions may change the limit only if they are able to interact effectively with *all* sites in Λ. When this happends we say that a short range interaction becomes effectively long range.

The analysis of the large volume limit can then be translated in the analysis of multiscale effects (short range interactions becoming effectively long range).

Note that, except in some special cases, we cannot compute the integrals in a closed form! We need tools to get estimates as precise as possible.

1.6 Plan of the course

We will consider some examples of functional integrals arising from statistical mechanics and physics, and learn techniques to construct the limit as the number of variables tends to infinity.

In models coming from statistical mechanics the measure is always a probability measure (real positive and normalized to 1). The field φ may take values in a discrete set (ex: ± 1), in a bounded set (ex: $\varphi(x) = \cos(\theta_x)$, with $\theta \in [0, 2\pi[$), in an unbounded set $\varphi(x) \in \mathbb{R}$.

In models coming from physics, the measure may become complex valued, though still normalized to 1. The field φ may be a real or complex vector, a matrix and some components of φ may even be Grassmann variables (anticommuting numbers $ab = -ba$).

In many cases the energy is of the form

$$H(\varphi) = \sum_{x,y} J_{xy} \|\varphi(x) - \varphi(y)\|^2 + \sum_x V(\varphi(x)),$$

where $J_{xy} = J_{yx} \geqslant 0$. The first term creates an interaction between different sites, the second term gives a set of independent constraints on each variable (it is called the diagonal term). When $J_{xy} = 0\ \forall x, y$, the measure $e^{-\beta H}$ factors in a product of measures. When $V = 0$, the integral cannot be factored. Depending on the relative size of the parameters, we will see that the integral is dominated by the interaction term or the diagonal term. These two situations correspond to different physical properties in the underlying model.

Some examples of potential V are

- $V(\varphi) = m^2\|\varphi\|^2 + \lambda\|\varphi\|^4$ (single well)
- $V(\varphi) = \lambda(\|\varphi\|^2 - \mu)^2$ (double well or mexican hat)
- $V(\varphi) = \lambda \ln(1 + \|\varphi\|^2)$ (log potential)

In the first two cases the potential is a convex function, for large $\|\varphi\|$, in the last case, the function becomes concave, adding additional problems.

1.7 Digression: why choosing an exponential weight?

We will motivate the choice of an exponential weight in the simpler case of a *finite* set of possible configurations. On can justify the same arguments in the general case (see for instance the lecture notes by Stefan Adams, chapter 7, http://www.mis.mpg.de/preprints/ln/lecturenote-3006.pdf).

Let Ω_Λ the set of all possible configurations for our finite system (the cristal in our example) and let $H : \Omega \to \mathbb{R}$ be the corresponding energy functional. Since Ω is finite, in order to define a probability measure we only need to give a set of numbers $\{\mu(\omega)\}_{\omega\in\Omega}$ such that $0 \leqslant \mu(\omega) \leqslant 1$ and $\sum_\omega \mu(\omega) = 1$. Let us consider how the system is connected to the external world.

Case 1: Isolated system The only way to change the energy of the system is to "give away" some of it or "take in " some of it from outside. But an isolated system has no exchange with the exterior so in this case the energy is fixed $H(\omega) = E \ \forall\omega \in \Omega$. Then there is no way to decide which configuration is preferable (they all have the same energy) and the most reasonable choice for μ is the uniform distribution that assigns the same weight to each configuration:

$$\mu(\omega) = \frac{1}{|\Omega|} \qquad \forall\omega \in \Omega, \quad \text{where}|\Omega| = \text{cardinal of } \Omega.$$

Case 2: system in contact with a reservoir at fixed temperature T In this case the energy can change, but since the temperature outside is fixed the *average energy* of our system is given by

$$\mathbb{E}[H] = \sum_{\omega\in\Omega} \mu(\omega)H(\omega) = E(T). \tag{1.7.3}$$

This time a uniform measure would not work since we expect large deformations (large energies) to be more unlikely than small deformations (small energies). The correct choice is to take a measure "as uniform as possible" under the constraint (1.7.3). To quantify how "uniform" a measure is we use the entropy. Let $\mathcal{M}(\Omega)$ be the set of probability measures on Ω, then the entropy is defined as

$$\begin{aligned} S : \quad & \mathcal{M}(\Omega) \to \mathbb{R} \\ & \mu \to S(\mu) = -\textstyle\sum_\omega \mu(\omega) \ln \mu(\omega). \end{aligned}$$

To see the kind of information we obtain from S let us consider two extreme cases:

(a) $\mu(\omega) = 1/|\Omega|$ the uniform measure. Then we have the same propability of being anywhere inside Ω: this means we have as little information as possible. In this case $S(\mu) = \ln(|\Omega|)$, that is a large number when Ω is large.

(b) $\mu(\omega) = \delta_{\omega\omega_0}$ a measure localized on just one element ω_0 of Ω. Then we know (with probability 1) that we must be exactly on the configuration ω_0: this means we have the maximal information. In this case $S(\mu) = -\mu(\omega_0)\ln(\omega_0) = 0$.

In general, the more "uniform" our measure is, the larger S. Therefore we choose the measure μ that maximizes $S(\mu)$, under the constraint (1.7.3).

Lemma 1 *Let H be a (non constant) energy functional,*

$$\nu_\beta(\omega) = \frac{e^{-\beta H(\omega)}}{Z_\beta}, \qquad Z_\beta = \sum_\omega e^{-\beta H(\omega)}.$$

a Gibbs measure and set

$$\begin{array}{lll} f: &]0,+\infty] & \to \mathbb{R} \\ & \beta & \to f(\beta) = \mathbb{E}_{\nu_\beta}[H]. \end{array}$$

Then we have the following results.

(a) For each $E \in Range(f)$ there exists a parameter $\bar\beta$ such that for $\beta = \bar\beta$ $\mathbb{E}_{\nu_{\bar\beta}}[H] = E$.

(b) For any probability measure satisfying $\mathbb{E}_\mu[H] = E$ we have $S(\mu) \leqslant S(\nu_{\bar\beta})$. Equality holds only for $\mu = \nu_{\bar\beta}$.

Remark When H is constant ($H(\omega) = H \ \forall\omega$) then the Gibbs measure ν_β coincides with the uniform measure for any choice of β.

Proof Let $\bar H_\beta = \mathbb{E}_{\nu_\beta}[H]$. In order to prove (a) note that

$$f'(\beta) = -\mathbb{E}_{\nu_{\bar\beta}}\left[(H - \bar H_\beta)^2\right] = -\sum_\omega \nu_\beta(\omega)(H(\omega) - \bar H_\beta)^2 < 0,$$

since $\nu_\beta(\omega) > 0 \ \forall\omega$ and there is at least one ω where $(H(\omega) - \bar H_\beta)^2 > 0$ (otherwise H would be the constant function). Then f is injective, hence (a).

To prove (b), note that

$$S(\mu) = -\sum_\omega \mu(\omega)\ln\mu(\omega) = -\sum_\omega \mu(\omega)\ln\frac{\mu(\omega)}{\nu_{\bar\beta}(\omega)} - \sum_\omega \mu(\omega)\ln\nu_{\bar\beta}(\omega).$$

Now using the definition of ν_β the second term is

$$-\sum_\omega \mu(\omega)\ln\nu_{\bar\beta}(\omega) = \sum_\omega \mu(\omega)\ln Z_{\bar\beta} + \bar\beta\sum_\omega \mu(\omega)H(\omega) = \ln Z_{\bar\beta} + \bar\beta E = S(\nu_{\bar\beta})$$

Inserting this we have

$$S(\mu) = S(\nu_{\bar\beta}) - \sum_\omega \mu(\omega)\ln\frac{\mu(\omega)}{\nu_{\bar\beta}(\omega)} = S(\nu_{\bar\beta}) - \sum_\omega \nu_{\bar\beta}(\omega)\Phi(X(\omega)) = S(\nu_{\bar\beta}) - \mathbb{E}_{\nu_{\bar\beta}}[\Phi(X(\omega))]$$

where we set

$$\Phi(x) = x\ln x, \qquad X(\omega) = \frac{\mu(\omega)}{\nu_{\bar\beta}(\omega)}.$$

Now $\Phi''(x) = 1/x > 0$ so by Jensen's inequality

$$\mathbb{E}_{\nu_{\bar\beta}}[\Phi(X(\omega))] \geqslant \Phi\left(\mathbb{E}_{\nu_{\bar\beta}}[X(\omega)]\right) = \Phi(1) = 0.$$

Since Φ is strictly convex, equality holds only when $X(w)$ is a constant function, that means there exist a constant K such that $\mu(\omega) = K\nu_{\bar\beta}(\omega) \ \forall\omega$. But $\sum_\omega \mu(\omega) = \sum_\omega \nu_{\bar\beta}(\omega) = 1$, then $K = 1$. This completes the proof of (b). □

Chapter 2

One dimensional problems

When $d = 1$ our finite region Λ is a finite chain of points $\Lambda = (-L, -L + 1, \dots 0, 1, \dots, L)$. The techniques applying to 1d systems can be generalized to quasi-one dimensional systems, such as strips of finite width. The material in this chapter is mostly based on the lecture notes by A. Kupiainen [?] (for the Ising model part) and on the book by B. Helffer [?] (for the part on integral operators).

2.1 Ising model

We will define the model in general dimension and later specialize to $d = 1$. Let $\bar{L} = (L_1, \dots L_d) \in \mathbb{N}^d$ and $\Lambda = [-L_1, \dots, L_1] \times \cdots [-L_d, \dots L_d]$ a rectangle in $\mathbb{Z}^d$ centered around the origin. To each site $x \in \Lambda$ we associate a spin (the analog of $\varphi(x)$ in the cristal example) taking only two values $-1, +1$. The configuration space is then $\Omega_\Lambda = \{1, -1\}^\Lambda$ and a configuration of the finite system is

$$\begin{array}{lll} \sigma: & \Lambda & \to \{1,-1\}^\Lambda \\ & x & \to \sigma(x), \end{array}$$

where $\sigma(x)$ is called the "spin" at site x. Let $\Omega = \{1, -1\}^{\mathbb{Z}^d}$ be the set of spin configurations on the whole lattice.

The energy for a configuration $\sigma \in \Omega_\Lambda$, is given by the finite volume Ising Hamiltonian $H_\Lambda^{\bar{\sigma}} : \Omega_\Lambda \to \mathbb{R}$

$$H_\Lambda^I(\sigma) = -J \sum_{x \sim y \in \Lambda} \sigma_x \sigma_y - \sum_{x \in \Lambda} h \sigma_x, \qquad J > 0, \; h \in \mathbb{R}$$

The first term in H^I represents an interaction between nearest neighbor sites and the parameter J is called the coupling constant. The last term is a sum of independent contributions at each site. The parameter h is called the external magnetic field.

Phenomenology The coupling term in H^I is minimum when all spins σ_x have the same orientation $\sigma_x = +1\ \forall x$, or $\sigma_x = -1\ \forall x$: in both cases the coupling contribution is $-J\sum_{x\sim y\in\Lambda} 1$. On the other hand, the second sum in H^I is minimum when all spins have the same sign as h, hence for $h \neq 0$ the only spin configuration minimizing the energy is $\sigma_x = \text{sign}(h)\ \forall x \in \Lambda$. In this sense, h plays the role of an external magnetic field for a ferromagnetic material: when $h = 0$ the spins try to align, but since they do not know which direction to take ($+1$ or -1) they end up being half $+1$ and half -1 so the average orientation is zero. When an external field h is present, the spins align with it.

Note that when $J < 0$ nearest neighbor spin pairs try to take *opposite* spin orientations. This is called paramagnetic behavior.

History The Ising model was introduced to describe ferromagnetic materials, but it proved to be relevant in a wide variety of problems, from lattice gases, to biology, economics and image analysis.

2.1.1 Boundary conditions

Let $\bar{\sigma} \in \{1, -1\}^{\mathbb{Z}^d}$ a *fixed* configuration on the *infinite* lattice.

Definition 1 The boundary of Λ is defined by

$$\partial\Lambda = \{x \in \Lambda |\, \exists y \in \Lambda^c \text{with } \|x - y\| = 1\}$$

Definition 2 The Ising Hamiltonian with $\bar{\sigma}$ boundary conditions is $H_\Lambda^{\bar{\sigma}}$: $\Omega_\Lambda \to \mathbb{R}$

$$H_\Lambda^{\bar{\sigma}}(\sigma) = H^I(\sigma) - J \sum_{x\in\partial\Lambda} \sum_{y\in\Lambda^c, y\sim x} \sigma_x \bar{\sigma}_y$$

where $\bar{\sigma} \in \Omega$ is some fixed infinite volume spin configuration.

The Ising Hamiltonian with periodic boundary conditions is $H_\Lambda^{per} : \Omega_\Lambda \to \mathbb{R}$

$$H_\Lambda^{per}(\sigma) = -J \sum_{x\sim y\in\mathbb{T}_{\bar{\mathbb{L}}}} \sigma_x\sigma_y - h\sum_{x\in\Lambda} \sigma_x$$

where $T_{\bar{L}} = \mathbb{Z}/L_1 \times \cdots \times \mathbb{Z}/L_d$ is a torus.

Finally The Ising Hamiltonian with free boundary conditions is

$$H_\Lambda^{free}(\sigma) = H_\Lambda^I(\sigma).$$

2.1.2 Probability measure and thermodynamic limit

Let $H^{(bc)_\Lambda}$ be the finite volume Ising energy with some fixed boundary conditions. We define a probability measure on Ω_Λ by

$$\mu_{\Lambda,\beta}^{(bc)}(\sigma) = \frac{e^{-\beta H_\Lambda^{(bc)}(\sigma)}}{Z_{\Lambda,\beta}^{(bc)}}$$

where the normalization factor

$$Z^{(bc)}_{\Lambda,\beta} = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(bc)}_\Lambda(\sigma)}$$

is called the partition function. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be a growing sequence of regions s.t. $\Lambda_n \subset \Lambda_{n+1}$ $\forall n$ and $\lim_{n\to\infty} \Lambda_n = \mathbb{Z}^d$.

We denote by $\mathcal{F}_n$ the sigma algebra on Ω_{Λ_n} and by $\mathcal{F}$ the (infinite volume) sigma algebra on Ω. Then each measure $\mu^{(bc)}_{\Lambda_n,\beta}$ on $\mathcal{F}_n$ can be extended to a measure $\tilde{\mu}_n$ on $\mathcal{F}$ with the following definition

$$\begin{aligned} \tilde{\mu}_n(A) &= 0 && \text{if } A \cap \Omega_{\Lambda_n} = \emptyset \\ &= \mu^{(bc)}_{\Lambda_n,\beta}(A \cap \Omega_{\Lambda_n}) && \text{otherwise.} \end{aligned}$$

In order to study the thermodynamic limit we will consider the following class of functions.

Definition: local functions. A function $f : \Omega \to \mathbb{R}$ is local if it depends only on the spin value on a finite set of lattice points. Precisely, f is local if $\exists$ a set $X \subset \mathbb{Z}^d$ with $|X_f| < \infty$ and a function $F : \Omega_X \to \mathbb{R}$ s.t.

$$f(\sigma) = F(\sigma_X) \qquad \forall \sigma \in \Omega,$$

where $\sigma_X = \{\sigma_x\}_{x\in X}$ is the restriction of the configuration σ to the set X.

Example The functions $f_1(\sigma) = \sigma_{x_1}$ and $f_2(\sigma) = \sigma_{x_1}\sigma_{x_2}$ (where x_1, x_2 are fixed lattice points) are both local functions with $X = \{x_1\}$, $\{x_1, x_2\}$ respectively. We will see below that all local functions can be obtained from functions of this form.

Lemma For any function $f : \Omega \to \mathbb{R}$ depending only on spins inside the finite set X, there exists a family of real parameters $\{a_A\}_{A\subseteq X}$ associated to each subset of X satisfying

$$f(\sigma) = \sum_{A\subseteq X} a_A \sigma_A$$

where

$$\sigma_A = \prod_{x\in A} \sigma_x.$$

Proof. Let $\mathbf{1}_+(\sigma_x) = \mathbf{1}_{\{\sigma_x=1\}}(\sigma_x)$ and $\mathbf{1}_-(\sigma_x) = \mathbf{1}_{\{\sigma_x=-1\}}(\sigma_x)$. This can be written in the more condensed form

$$\mathbf{1}_{\sigma'_x}(\sigma_x) = \delta_{\sigma_x,\sigma'_x} = \mathbf{1}_{\sigma_x}(\sigma'_x), \qquad \sigma_x, \sigma'_x = \pm 1.$$

Let $\chi_1 = (\mathbf{1}_+ + \mathbf{1}_-)/2$ and $\chi_2 = (\mathbf{1}_+ - \mathbf{1}_-)/2$. Then

$$\mathbf{1}_\sigma = \chi_1 + \sigma\chi_2, \qquad \text{with } \sigma = \pm 1.$$

and the function

$$\begin{aligned} \mathbf{1}_{\sigma'}(\sigma) &= \prod_{x\in X} \mathbf{1}_{\sigma'_x}(\sigma_x) = \prod_{x\in X} \mathbf{1}_{\sigma_x}(\sigma'_x) = \prod_{x\in X} \left[\chi_1(\sigma'_x) + \sigma_x \chi_2(\sigma'_x)\right] \\ &= \sum_{A\subseteq X} \prod_{x\in A} \sigma_x \prod_{x\in A} \chi_2(\sigma'_x) \prod_{x\in x\setminus A} \chi_2(\sigma'_x) \end{aligned}$$

equals 1 when $\sigma = \sigma'$ and equals 0 otherwise. Then

$$\begin{aligned} f(\sigma) &= \sum_{\sigma'} \mathbf{1}_{\sigma'}(\sigma) f(\sigma) = \sum_{\sigma'} \mathbf{1}_{\sigma'}(\sigma) f(\sigma') \\ &= \sum_{\sigma'} f(\sigma') \prod_{x\in X} \left[\chi_1(\sigma'_x) + \sigma_x \chi_2(\sigma'_x)\right] \\ &= \sum_{A\subseteq X} \prod_{x\in A} \sigma_x \left\{ \sum_{\sigma'} f(\sigma') \prod_{x\in A} \chi_2(\sigma'_x) \prod_{x\in x\setminus A} \chi_2(\sigma'_x) \right\} \\ &= \sum_{A\subseteq X} \prod_{x\in A} \sigma_x a_X \end{aligned}$$

where a_X is a constant independent of the configuration σ. ■

Definition: thermodynamic limit We say that the sequence of measures $\mu^{(bc)}_{\Lambda_n,\beta}$ converges to a measure μ on Ω if

$$\mathbb{E}_{\tilde{\mu}_n}[f] = \sum_{\sigma} \tilde{\mu}_n(\sigma) f(\sigma) \to_{n\to\infty} \sum_{\sigma} \mu(\sigma) f(\sigma) = \mathbb{E}_{\mu}[f]$$

for all local functions $f : \Omega \to \mathbb{R}$.

By the lemma above, it is enough to prove the existence of the limit for $\mathbb{E}_{\mu}[\sigma_X]$ for any subset X with $|X| < \infty$.

2.2 Transfer matrix for the Ising model in one dimension

Let $\Lambda = [-L, \ldots, L]$. The finite volume Ising Hamiltonian in $d = 1$ can be written

$$H^I_\Lambda(\sigma) = -J \sum_{x=-L}^{L-1} \sigma_x \sigma_{x+1} - h \sum_{x\in\Lambda} h\sigma_x, \qquad J > 0,\ h \in \mathbb{R}$$

The boundary is reduced to two points $\partial\Lambda = \{-L, L\}$, therefore the Hamiltonian with $\bar{\sigma}$ (resp. periodic, free) boundary conditions is

$$\begin{aligned} H^{\bar{\sigma}}_\Lambda(\sigma) =& H^I(\sigma) - J\left[\sigma_{-L}\bar{\sigma}_{-L-1} + \sigma_L \bar{\sigma}_{L+1}\right] \\ H^{per}_\Lambda(\sigma) =& H^I(\sigma) - J\sigma_L\sigma_{-L} \\ H^{free}_\Lambda(\sigma) =& H^I_\Lambda(\sigma). \end{aligned}$$

where $\bar{\sigma} \in \Omega$ is some fixed infinite volume spin configuration.

2.2.1 Partition function

Let $Z^{(bc)}_{\Lambda,\beta}$ be the partition function at finite volume with some fixed boundary conditions. Then we have

Lemma 2 *The limit as $L \to \infty$ of $|\Lambda|^{-1} \ln Z^{(bc)}_{\Lambda,\beta}$ is finite and independent of the boundary conditions*

Proof We will prove this result for $\bar\sigma$, periodic and free boundary conditions. In the case of $\bar\sigma$ and periodic boundary conditions, the partition function can be written as

$$\begin{aligned} Z^{(bc)}_{\Lambda,\beta} = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(bc)}_\Lambda(\sigma)} &= \sum_{\sigma\in\Omega_\Lambda} F^{left}_h(\sigma_{-L}) \left[\prod_{x=-L}^{L-1} T_h(\sigma_x,\sigma_{x+1})\right] F^{right}_h(\sigma_L) \\ &= \left(F^{left}_h, T^{2L}_h F^{right}_h\right) \end{aligned}$$

where T_h is a 2×2 matrix

$$T_h = \begin{pmatrix} e^{\beta+h\beta} & e^{-\beta} \\ e^{-\beta} & e^{\beta-h\beta} \end{pmatrix}, \qquad T_h(\sigma,\sigma') = e^{\frac{\beta h\sigma}{2}} e^{\beta\sigma\sigma'} e^{\frac{\beta h\sigma'}{2}}, \tag{2.2.1}$$

while $F^{left/right}_h$ are 2 component vectors encoding the boundary conditions

$$\begin{aligned} &F^{left}_h(\sigma) = e^{\beta\sigma\bar\sigma_{-L-1}} e^{\frac{\beta h\sigma}{2}}, \qquad F^{right}_h(\sigma) = e^{\frac{\beta h\sigma}{2}} e^{\beta\sigma\bar\sigma_{L+1}} \qquad \text{for } \bar\sigma \text{ b.c.}, \\ &F^{left}_h(\sigma) = F^{right}_h(\sigma) = e^{\frac{\beta h\sigma}{2}} \qquad \text{for free b.c.} \end{aligned} \tag{2.2.2}$$

Finally $(\ ,\ )$ denotes the real euclidean scalar product . In the case of periodic boundary conditions

$$\begin{aligned} Z^{(per)}_{\Lambda,\beta} = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(per)}_\Lambda(\sigma)} &= \sum_{\sigma\in\Omega_\Lambda} \left[\prod_{x=-L}^{L-1} T_h(\sigma_x,\sigma_{x+1})\right] T_h(\sigma_L,\sigma_{-L}) \\ &= \mathrm{Tr}\; T^{2L+1}_h \end{aligned}$$

To study the large volume properties of the partition function then, we have to study a 2×2 matrix, reducing the problem from 2^{2L+1} to 2 spins only. The matrix T_h is real symmetric hence diagonalisable. The eigenvalues are

$$\begin{aligned} \lambda_1 &= e^\beta \cosh(\beta h) + \sqrt{[e^\beta \sinh(\beta h)]^2 + e^{-2\beta}}, \\ \lambda_2 &= e^\beta \cosh(\beta h) - \sqrt{[e^\beta \sinh(\beta h)]^2 + e^{-2\beta}}, \ 0 < \lambda_2 < \lambda_1. \end{aligned}$$

Let v_1, v_2 the corresponding normalized eigenvectors and P_1, P_2 are 2×2 matrices corresponding to the orthogonal projections on v_1, v_2:

$$P_1(\sigma,\sigma') = v_1(\sigma)v_1(\sigma'), \qquad P_1(v) = (v_1,v)v_1, \ \forall v \in \mathbb{R}^2.$$

The definition for P_2 is similar. Since they are orthogonal projections P_1, P_2 satify

$$P_1^2 = P_1,\ P_2^2 = P_2,\ P_1P_2 = P_2P_1 = 0.$$

Moreover, the eigenvector v_1 for the largest eigenvalue has the following additional property, that will be crucial for our proof:

$$v_1(\sigma) > 0\ \forall\sigma.$$

Indeed let $v_1 = (x_1, y_1)$. Then we obtain

$$y_1 = x_1 C_1 \qquad \text{where } C_1 = e^{\beta}\left[\sqrt{[e^{\beta}\sinh(\beta h)]^2 + e^{-2\beta}} - [e^{\beta}\sinh(\beta h)]\right]. \tag{2.2.3}$$

Since $C_1 > 0$ for any choice of β, h the two components x_1 and y_1 must have the same sign. Inserting the spectral decomposition $T = \lambda_1 P_1 + \lambda_2 P_2$ in the expression for Z we have

$$Z^{(bc)}_{\Lambda,\beta} = \left(F_h^{left}, T_h^{2L} F_h^{right}\right) = \lambda_1^{2L}\left[\left(F_h^{left}, P_1 F_h^{right}\right) + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L}\left(F_h^{left}, P_2 F_h^{right}\right)\right]$$
$$= \lambda_1^{2L}\left[\left(F_h^{left}, v_1\right)\left(v_1, F_h^{right}\right) + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L}\left(F_h^{left}, P_2 F_h^{right}\right)\right]$$

To complete the proof we need two ingredients

- the first term in the parenthesis is strictly positive. Indeed $(F_h^{left}, v_1) = \sum_\sigma F_h^{left}(\sigma) v_1(\sigma) > 0$ since $v_1(\sigma) > 0$ and $F^{left}(\sigma) > 0$ for all σ. For the same reason $(v_1, F_h^{right}) > 0$.
- the second term in the parenthesis disappears in the limit $L \to \infty$. This holds since $|\lambda_2| < \lambda_1$.

Using these two ingredients we obtain

$$\frac{\ln Z^{(bc)}_{\Lambda,\beta}}{2L+1} = \tfrac{2L}{2L+1}\ln\lambda_1 + \tfrac{1}{2L+1}\ln\left[(F_h^{left}, v_1)(v_1, F_h^{right}) + \left(\tfrac{\lambda_2}{\lambda_1}\right)^{2L}(F_h^{left}, P_2 F_h^{right})\right]$$
$$\to_{L\to\infty} \ln\lambda_1 = \ln\left[e^{\beta}\cosh(\beta h) + \sqrt{[e^{\beta}\sinh(\beta h)]^2 + e^{-2\beta}}\right]$$

Since the boundary conditions appear only in $F^{left/right}$, the result is the same for free, or for any choice of $\bar\sigma$ boundary conditions.

In the case of periodic boundary conditions

$$Z^{(per)}_{\Lambda,\beta} = \text{Tr } T_h^{2L+1} = \lambda_1^{2L+1}\text{Tr } P_1 + \lambda_2^{2L+1}\text{Tr } P_2 = \lambda_1^{2L+1}\left[1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L+1}\right]$$

Therefore

$$\frac{\ln Z^{(per)}_{\Lambda,\beta}}{2L+1} = \ln \lambda_1 + \tfrac{1}{2L+1} \ln \left[1 + \left(\tfrac{\lambda_2}{\lambda_1}\right)^{2L+1} \right] \to_{L\to\infty} \ln \lambda_1 .$$

The limit exists for any choice of β, h and coincides with the result obtained with free or $\bar{\sigma}$ boundary conditions. □

2.2.2 Average magnetization

The finite volume average magnetization at position x is defined by

$$\mathbb{E}_\Lambda[\sigma_x] = \frac{\sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(bc)}_\Lambda(\sigma)}\sigma_x}{\sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^{(bc)}_\Lambda(\sigma)}}$$

We have the following result

Lemma 3 *The average magnetization has a limit*

$$\mathbb{E}_\Lambda[\sigma_x] \to_{L\to\infty} M_\beta(h) = \frac{1-C_1^2}{1+C_1^2} \tag{2.2.4}$$

where C_1 *is given in* (2.2.3). *The limit* $M_\beta(h)$ *is independent of* x *and the boundary conditions, is a smooth increasing function of* h *satisfying*

$$\begin{aligned} &-1 < M_\beta(h) < +1 \ \forall h \in \mathbb{R}, \\ &\lim_{h\to\infty} M_\beta(h) = +1, \ \lim_{h\to-\infty} M_\beta(h) = -1, \end{aligned}$$

and has the same sign as h. *In particular* $M_\beta(0) = 0$.

Remark 1 This result is consistent with the physical intuition saying that the spins try to align with the magnetic field h. When h becomes very large all spins align hence the magnetization becomes $+1$ (resp. -1) depending if $h > 0$ or $h < 0$.

Remark 2 The function $M : \mathbb{R} \to]-1,1[$ is invertible, so we could use the magnetization M as a parameter in our measure instead of h: $\mu_{\beta,h(M)}$,

Proof For simplicity we consider $x > 0$. The same arguments then hold for $x \leqslant 0$.

As in the case of the partition function we can express $\mathbb{E}_\Lambda[\sigma_x]$ in terms of the transfer matrix T_h:

$$\mathbb{E}_\Lambda[\sigma_x] = \frac{\left(F^{left}_h, T^{L+x}_h \ \Sigma \ T^{L-x}_h F^{right}_h\right)}{\left(F^{left}_h, T^{2L}_h F^{right}_h\right)}$$

where $T_h, F_h^{left}, F_h^{right}$ were defined in (2.2.1) and (2.2.2) above. The 2×2 matrix Σ encodes the new term σ_x

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Sigma_{\sigma,\sigma'} = \delta_{\sigma\sigma'}\sigma.$$

Inserting the spectral decomposition $T = \lambda_1 P_1 + \lambda_2 P_2$ we get

$$\mathbb{E}_\Lambda[\sigma_x] = \frac{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L+x} P_2\right] \Sigma \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L-x} P_2\right] F_h^{right}\right)}{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L} P_2\right] F_h^{right}\right)}$$

$$\to_{L\to\infty} \frac{\left(F_h^{left}, P_1 \ \Sigma \ P_1 F_h^{right}\right)}{\left(F_h^{left}, P_1 F_h^{right}\right)} = \frac{(F_h^{left}, v_1)(v_1, \ \Sigma v_1)(v_1, F_h^{right})}{(F_h^{left}, v_1)(v_1 F_h^{right})}$$

$$= (v_1, \ \Sigma v_1) = \sum_\sigma v_1(\sigma)^2 \sigma = v_1(+1)^2 - v_1(-1)^2,$$

where we used as before $(F_h^{left}, v_1) > 0$, $(F_h^{right}, v_1) > 0$ and $|\lambda_2| < \lambda_1$. Using (2.2.3) we see that

$$v_1(1)^2 - v_1(-1)^2 = \frac{1 - C_1^2}{1 + C_1^2}$$

where $C_1 > 0$ is a smooth function of h and satisfies

$$C_1 < e^\beta [e^{-\beta}] = 1 \qquad \text{when } h > 0$$
$$C_1 > e^\beta [e^{-\beta}] = 1 \qquad \text{when } h < 0$$
$$C_1 = 1 \qquad \text{when } h = 0.$$

Therefore $M(h)$ has the same sign as h and $M(0) = 0$. Moreover

$$C_1' = e^{2\beta} \beta \cosh(\beta h) \left[\frac{e^\beta \sinh(\beta h)}{\sqrt{(e^\beta \sinh(\beta h))^2 + e^{-2\beta}}} - 1 \right] < 0 \qquad \forall h,$$

then $M'(h) = -\frac{4C_1 C_1'}{(1+C_1^2)^2} > 0 \ \forall h$. Finally

$$C_1(h) = e^{2\beta} \sinh(\beta h) \left[\sqrt{1 + \frac{e^{-4\beta}}{\sinh^2(\beta h)}} - 1 \right] = O\left(\frac{1}{\sinh(\beta h)}\right) \to_{h\to\infty} 0$$

$$C_1(h) = e^{2\beta} |\sinh(\beta h)| \left[2 + O\left(\frac{1}{\sinh^2(\beta h)}\right) \right] \to_{h\to-\infty} +\infty$$

hence $\lim_{h\to\pm\infty} M(h) = \pm 1$. This completes the proof. □

2.2.3 Spin-spin correlation

The two spin correlation is defined by

$$C_{xy}^\Lambda = \mathbb{E}_\Lambda[\sigma_x \sigma_y] - \mathbb{E}_\Lambda[\sigma_x] \mathbb{E}_\Lambda[\sigma_y].$$

This quantity is zero when σ_x and σ_y are independent. We have the following result

Lemma 4 *The infinite volume limit for C^Λ_{xy} exists, is independent of the boundary conditions and satisfies*

$$\lim_{L\to\infty} C^\Lambda_{xy} = C_{xy} \;\; = \;\; Ke^{-\frac{|x-y|}{\xi}}$$

where $\xi > 0$, $K \geqslant 0$ are constants independent of x and y. The parameter ξ gives the distance where the spin correlation starts to become small and is called the localization distance.

Proof As in the previous subsections we use the transfer matrix representation. Without loss of generality we can consider $y > x$. Then

$$\mathbb{E}_\Lambda[\sigma_x\sigma_y] = \frac{\left(F_h^{left}, T_h^{L+x}\;\Sigma\; T_h^{y-x}\;\Sigma\; T_h^{L-y} F_h^{right}\right)}{\left(F_h^{left}, T_h^{2L} F_h^{right}\right)}$$

where $F_h^{left/right}$, T_h and Σ are defined above. Inserting the spectral decomposition $T = \lambda_1 P_1 + \lambda_2 P_2$ we get

$$\mathbb{E}_\Lambda[\sigma_x\sigma_y] = \frac{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L+x} P_2\right]\;\Sigma\;\left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{y-x} P_2\right]\;\Sigma\;\left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{L-y} P_2\right] F_h^{right}\right)}{\left(F_h^{left}, \left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{2L} P_2\right] F_h^{right}\right)}$$

$$\to_{L\to\infty} \frac{\left(F_h^{left}, P_1\;\Sigma\;\left[P_1 + \left(\frac{\lambda_2}{\lambda_1}\right)^{y-x} P_2\right]\;\Sigma\; P_1 F_h^{right}\right)}{\left(F_h^{left}, P_1 F_h^{right}\right)}$$

$$= \frac{(F_h^{left}, P_1\;\Sigma\;P_1\;\Sigma\;P_1F_h^{right})}{(F_h^{left}, P_1F_h^{right})} + \left(\frac{\lambda_2}{\lambda_1}\right)^{y-x} \frac{(F_h^{left}, P_1\;\Sigma\;P_2\;\Sigma\;P_1F_h^{right})}{(F_h^{left}, P_1F_h^{right})}$$

The first term in this sum gives

$$\frac{(F_h^{left}, P_1\;\Sigma\;P_1\;\Sigma\;P_1F_h^{right})}{(F_h^{left}, P_1F_h^{right})} = \frac{(F_h^{left}, v_1)(v_1,\Sigma v_1)(v_1,\Sigma v_1)(v_1, F_h^{right})}{(F_h^{left}, v_1)(v_1, F_h^{right})}$$
$$= (v_1, \Sigma v_1)^2 = M_\beta(h)^2 = \lim_{L\to\infty} \mathbb{E}_\Lambda[\sigma_x]\;\mathbb{E}_\Lambda[\sigma_y].$$

Therefore $\lim_{L\to\infty} C^\Lambda_{xy} = Ke^{-|x-y|/\xi}$ with

$$K = \frac{(F_h^{left}, P_1\;\Sigma\;P_2\;\Sigma\;P_1F_h^{right})}{(F_h^{left}, P_1F_h^{right})} = (v_1, \Sigma v_2)^2, \qquad \xi = \frac{1}{\ln\frac{\lambda_1}{\lambda_2}}.$$

The values of K and ξ do not depend on F_h so the result is the same for all boundary conditions. Similar arguments hold in the case of periodic boundary conditions.

Comparison with the case of no interaction If we set $J = 0$ instead of $J = 1$ in the Ising Hamiltonian we obtain a product measure on Ω_Λ

$$\mu_{J=0}(\sigma) = \prod_{j=-L}^{L-1} e^{\beta h \sigma_x},$$

and all correlation functions are easy to compute. In particular

$$\begin{aligned} Z_\Lambda^{J=0} &= \prod_{j=-L}^{L-1} \sum_{\sigma_x \in \{1,-1\}} e^{\beta h \sigma_x} = [2\cosh(\beta h)]^{2L+1} \\ \mathbb{E}_{\Lambda,J=0}[\sigma_x] &= \frac{\sum_{\sigma_x \in \{1,-1\}} e^{\beta h \sigma_x} \sigma_x}{\sum_{\sigma_x \in \{1,-1\}} e^{\beta h \sigma_x}} = \frac{\sinh(\beta h)}{\cosh(\beta h)} = M(h) \\ C_{\Lambda,J=0}^{xy} &= 0 \qquad \forall x, y, \ \forall \Lambda. \end{aligned}$$

As in the $J = 1$ case the magnetization $M(h)$ is invertible, and we can define h as a function of M, i.e. the magnetization we want to obtain: $h(M) = \frac{1}{\beta} \tanh^{-1}(M)$. All correlation functions are zero because the measure is factored over a product of local measures. The infinite volume measure exists and is given by

$$\mu_{\beta,M}^{J=0}(\sigma) = \prod_{x \in \mathbb{Z}} e^{\beta h_0(M) \sigma_x}, \qquad h_0(M) = \tfrac{1}{\beta} \tanh^{-1}(M).$$

In the case $J = 1$, we have see that two point correlations decay exponentially and one can show the same result for all correlation functions. This means that the infinite volume measure $\mu_{\beta,J=1}$ is "approximately" the product measure (in a sense to be made precise)

$$\mu_{\beta,J=1} \sim \prod_{x \in \mathbb{Z}} e^{\beta h_1(M) \sigma_x}, \qquad h_1(M) = M_{\beta,J=1}^{-1}(M),$$

where the magnetization h_1 is now fixed by the function (2.2.4). Therefore the measure "looks like" what we get in the $J = 0$ case, with a modified parameter h. We say the magnetic field parameter has been "renormalized".

2.2.4 Generalizations: transfer matrix in a strip

Let $\Lambda = \{-L, \dots, L\} \times \{1, \dots, W\}$. When $L \to \infty$ this becomes an infinite strip. Its properties are similar to 1d chain, hence this is called a "quasi-one dimensional" problem. A point $\vec{x} \in \Lambda$ is identified by two coordinates $\vec{x} = (x, y)$ with $x \in \{-L, \dots, L\}$, $y \in \{1, \dots, W\}$. The space of configurations is $\Omega_\Lambda = \{1, -1\}^\Lambda$ and the Ising Hamiltonian on the strip is

$$\begin{aligned} H^I(\sigma) &= -J \sum_{\vec{x} \sim \vec{y} \in \Lambda} \sigma_{\vec{x}} \sigma_{\vec{y}} - h \sum_{\vec{x} \in \Lambda} \sigma_{\vec{x}} \\ &= -J \sum_{x=-L}^{L-1} \left[\sum_{y=1}^{W} \sigma_{x,y} \sigma_{x+1,y} \right] - \sum_{x=-L}^{L} \left[J \sum_{y=1}^{W-1} \sigma_{x,y} \sigma_{x,y+1} + h \sum_{y=1}^{W} \sigma_y \right] \end{aligned}$$

where in the first term we have the (horizontal) interactions between spins at the same height y, and in the second term we put together all terms involving only spins on the same vertical line corresponding to x. To make the transfer matrix easier to see, we define

$$X_x(y) = \sigma_{x,y}, \ y \in \{1, \dots, W\}$$

the vector made with all spins on the vertical line x. The configuration σ can be written in terms of X

$$\sigma = \{\sigma_{x,y}\}_{(x,y)\in\Lambda} = \{X_x\}_{x=-L}^{L}$$

and we can write H^I as

$$H^I(\sigma) = H^I(X) = \sum_{x=-L}^{L-1} I(X_x, X_{x+1}) + \sum_{x=-L}^{L} D(X_x)$$

where the interaction I and the diagonal D terms are

$$I(X, X') = -J\sum_{y=1}^{W} X(y)X'(y), \qquad D(X) = -J\sum_{y=1}^{W-1} X(y)X(y+1) - h\sum_{y=1}^{W} X(y).$$

Then the partition function can be written as

$$Z_\Lambda = \sum_{\sigma\in\Omega_\Lambda} e^{-\beta H^I(\sigma)} = \sum_{X(-L),\dots X(L)} F^{left}(X_{-L}) \left[\prod_{x=-L}^{L-1} T(X_x, X_{x+1})\right] F^{right}$$
$$= (F^{left}, T^{2L} F^{right}).$$

where $F^{left}(X) = F^{right}(X) = e^{-\frac{\beta}{2}D(X)}$ and

$$T(X, X') = e^{-\frac{\beta}{2}D(X)} e^{-\beta I(X,X')} e^{-\frac{\beta}{2}D(X')}.$$

Instead of a 2×2 matrix this time we have a $2^W \times 2^W$ matrix and computing the eigenvalues and eigenvectors may become cumbersome. To avoid doing the explicit we apply the following result

Theorem 1 (Perron-Frobenius) *[without proof] Let T be a $N \times N$ real matrix with $T_{ij} > 0$ $\forall i, j$. Then*

1. *$\lambda = \|T\|$ is an eigenvalue of T*
2. *for any eigenvalue $\lambda' \neq \lambda$ we have $|\lambda'| < \lambda$,*
3. *λ is simple and the corresponding eigenvector can be chosen so that $v_j > 0$ $\forall j$.*
4. *let v be an eigenvector for $\lambda' \neq \lambda$. Then v must have some negative or zero components.*

In our case T is a real symmetric matrix, hence there exists a orthonormal basis of eivenvectors. Morevoer $T(X,X')>0\ \forall X,X'$ so the theorem ensures that the top eigenvalue (in absolute value) λ_1 is positive, simple and the corresponding eigenvector v_1 satisfies $v_1(j)>0\ \forall j$. Then

$$T=\sum_{j=1}^{2^W}\lambda_j P_j, \qquad T^{2L}=\lambda_1^{2L}\left[P_1+\sum_{j=2}^{2^W}\left(\frac{\lambda_j}{\lambda_1}\right)^{2L}P_j\right]$$

where P_j are orthogonal projections and $|\lambda_j|/\lambda_1<1\ \forall j\geqslant 2$. Then

$$\begin{aligned}\tfrac{1}{|\Lambda|}\ln Z_\Lambda &= \tfrac{2L}{W(2L+1)}\ln\lambda_1+\tfrac{1}{|\Lambda|}\left[(F^{left},P_1F^{right})+\sum_{j=2}^{2^W}\left(\tfrac{\lambda_j}{\lambda_1}\right)^{2L}(F^{left},P_j,F^{right})\right]\\ &\to_{L\to\infty}\tfrac{\ln\lambda_1}{W}\end{aligned}$$

since $(F^{left},P_1F^{right})=(F^{left},v_1)(v_1,F^{right})>0$. The magnetization and correlation functions can be studied in a similar way.

Remark The argument works since W is kept fixed while $L\to\infty$. If we try to send W to infinity at the same time several problems appear. Among them: *(a)* the ratio $|\lambda_j|/\lambda_0$ depends on W and may converge to 1, *(b)* the size of the matrix T diverges and we have to ensure the sum over orthogonal projections remains well defined. Far from being just a nuisance, these problems signal that something fundamentally different may happen in higher dimensions.

2.3 Transfer matrix for continuous spin

Let us now go back to the first example we gave in Ch. 1, namely the deformations inside a perfect cristal.

Let $\Lambda=\{-L,\dots,L\}$ as before. The spin $\sigma_x=\pm1$ at the position $x\in\Lambda$ is now replaced by the atom displacement $\phi_x\in\mathbb{R}$. The finite volume set of spin configurations $\{\sigma\in\{1,-1\}^\Lambda\}$ becomes now

$$\Omega_\Lambda=\mathbb{R}^\Lambda=\{\phi|\phi:\Lambda\to\mathbb{R}\}$$

We consider the energy functional

$$H_\Lambda(\phi)=\sum_{j=-L}^{L-1}[\phi_j-\phi_{j+1}]^2+\frac{m^2}{\beta}\sum_{j=-L}^{L}\phi_j^2$$

This corresponds to the hamiltonian (1.1.1) for a cristal in one dimension, with an additional term $m^2\sum_x\phi^2$, favoring configurations with ϕ_x near zero for each x. Intuitively, this means that each atom wants to remain near to its equilibrium position on the lattice, independently of what the other atoms do. The parameter $m > 0$ is called the mass, and we rescaled by β in order to simplify the formulas.

We will consider first the case of **free boundary conditions**: $H_\Lambda^{(free)} = H_\Lambda$. We define a probability measure

$$d\mu_\Lambda(\phi) = \frac{e^{-\beta H_\Lambda}}{Z_\Lambda} d\phi$$

where $d\phi = \prod_{j=-L}^{L} d\phi_j$ is the product Lebesgue measure and

$$Z_\Lambda = \int_{\mathbb{R}^{2L+1}} e^{-\beta H_\Lambda} d\phi = \int_{\mathbb{R}^{2L+1}} e^{-\beta \sum_{j=-L}^{L-1} [\phi_j - \phi_{j+1}]^2} e^{-m^2 \sum_{j=-L}^{L} \phi_j^2} d\phi$$

is the normalization constant. The integrand inside Z is strictly positive, so $Z > 0$. Moreover $[\phi_j - \phi_{j+1}]^2 \geqslant 0$ for any choice of ϕ then

$$0 < Z_\Lambda \leqslant \prod_{j=-L}^{L} \int_{\mathbb{R}} e^{-m^2 \phi_j^2} d\phi_j = \left(\sqrt{\frac{\pi}{m^2}} \right)^{2L+1} < \infty.$$

Hence the measure is well defined.

As we did in the Ising model, we start by studying $\ln Z/|\Lambda|$ as $\Lambda \to \mathbb{Z}$. Our goal is to mimick the strategy we developed in the Ising model. We can write Z_Λ as

$$Z_\Lambda = \int_{\mathbb{R}^{2L+1}} F^{left}(\phi_{-L}) \prod_{j=-L}^{L-1} k(\phi_j, \phi_{j+1})\ F^{right}(\phi_L) \tag{2.3.5}$$

where

$$k(\phi, \phi') = e^{-\frac{m^2}{2}\phi^2} e^{-\beta(\phi - \phi')^2} e^{-\frac{m^2}{2}\phi'^2} \qquad F^{left}(\phi) = F^{right}(\phi) = e^{-\frac{m^2}{2}\phi^2}. \tag{2.3.6}$$

This expression is identical to what we obtained in the Ising case, but sums are now replaced by integrals and the arguments we applied to not generalize automatically.

2.3.1 From matrices to integral kernels: transfer operator

In the Ising case we defined the transfer operator as

$$\begin{array}{lll} T: & \mathbb{R}^2 & \to \mathbb{R}^2 \\ & v & \to [Tv](\sigma) = \sum_{\sigma'} T_{\sigma\sigma'}\ v(\sigma') \end{array}$$

where T is a 2×2 matrix acting on $\mathbb{R}^2$ endowed with the norm $\|v\|^2 = \sum_\sigma v(\sigma)^2$. The natural generalization in this context is the integral operator

$$\begin{array}{lll} K: & L_2(\mathbb{R}) & \to L_2(\mathbb{R}) \\ & f & \to [Kf](\phi) = \int d\phi'\ k(\phi, \phi')\ f(\phi') \end{array} \tag{2.3.7}$$

where

$$k: \quad \mathbb{R}\times\mathbb{R} \to \mathbb{R}$$
$$(x,y) \to k(x,y)$$

is called the *integral kernel*. While the matrix operator T was trivially well defined, here we need to check that: *(a)* the function $k(\phi,\phi')f(\phi')$ is integrable and *(b)* the function kf is still in $L_2(\mathbb{R})$.

A simple criterion is given by the Schur's bound below.

Lemma 5 *[Schur's bound.] Let $k:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfy the two bounds*

$$M_1 = \sup_x \int_{\mathbb{R}} |k(x,y)|dy \ < \infty \tag{2.3.8}$$

$$M_2 = \sup_y \int_{\mathbb{R}} |k(x,y)|dx \ < \infty.$$

Then $Kf(x) = \int k(x,y)f(y)dy$ defines a bounded linear operator from $L_2(\mathbb{R})$ to $L_2(\mathbb{R})$, with

$$\|K\| \ \leqslant \ \sqrt{M_1 M_2} \tag{2.3.9}$$

Proof Let $f \in L_2(\mathbb{R})$. By Cauchy-Schwartz inequality

$$[Ff](x)^2 \leqslant \left[\int |k(x,y)||f(y)|dy\right]^2 = \int \sqrt{|k(x,y)|}\sqrt{|k(x,y)|}|f(y)|dy$$
$$\leqslant \left[\int |k(x,y)|dy\right]\left[\int |k(x,y)|f(y)^2 dy\right] \leqslant \ M_1 \int |k(x,y)|f(y)^2 dy$$

Using Fubini's theorem we have

$$\int dx \int dy |k(x,y)|f(y)^2 \ = \int dy \ f(y)^2 \int dx \ |k(x,y)| \leqslant M_2 \|f\|^2 < \infty.$$

As a consequence $\int |k(x,y)| \ f(y)^2 dy$ and hence also $\int |k(x,y)| \ |f(y)|dy$ exist for all x, (except eventually on sets of measure zero). Then $[Kf](x)$ is well defined and

$$\|Kf\|^2 \leqslant M_1 M_2 \|f\|^2$$

so $Kf \in L_2(\mathbb{R})$ and $\|K\| \leqslant \sqrt{M_1 M_2}$. □

Symmetric kernels When the kernel satisfies (2.3.8) and has the additional property $k(x,y) = k(y,x)$ we can write for any $f,g \in L_2(\mathbb{R})$

$$(f,Kg) = (Kf,g), \qquad \text{where } (f,g) = \int f(x)g(x)dx$$

is the real scalar product on $L_2(\mathbb{R})$.

In the case of the cristal the kernel given by (2.3.6)

$$k(x,y) = e^{-\frac{m^2}{2}x^2} e^{-\beta(x-y)^2} e^{-\frac{m^2}{2}y^2}$$

is symmetric and satisfies

$$M_1 = M_2 = \sup_x \int |k(x,y)| dy \leqslant \int e^{-\frac{m^2}{2} y^2} dy = \sqrt{\frac{2\pi}{m^2}} < \infty$$

Then K defines a symmetric bounded linear operator on $L_2(\mathbb{R})$ and we can write the partition function as

$$Z_\Lambda = (F^{left}, K^{2L} F^{rigth}). \tag{2.3.10}$$

2.3.2 Expanding in a sum of projections

In the Ising case we used the expansion $T = \lambda_1 P_1 + \lambda_2 P_2$, where P_1, P_2 are orthogonal projections. For an integral operator this decomposition in general does not exist. An integral operator "looks like" a finite matrix when it is compact. Precisely

Definition: compact operator. An operator $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is compact if it is the limit in norm of a sequence of finite rank operators, i.e. there exists a sequence $\{K_N\}_{N\in\mathbb{N}}$ such that $K_N : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, its image has finite dimension for each N and

$$\lim_{N\to\infty} \|K - K_N\| = 0.$$

There is an easy criterion to check if an operator is compact.

Criterion for compactness. If K is Hilbert-Schmidt then it is compact.

Definition: Hilbert-Schmidt operator. An operator $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is called Hilbert-Schmidt if the kernel satisfies

$$\int_{\mathbb{R}\times\mathbb{R}} |k(x,y)|^2 dxdy < \infty$$

In our example

$$\int_{\mathbb{R}\times\mathbb{R}} |k(x,y)|^2 dxdy = \int_{\mathbb{R}\times\mathbb{R}} e^{-m^2x^2} e^{-2\beta(x-y)^2} e^{-m^2y^2} dxdy$$
$$\leqslant \int_{\mathbb{R}} e^{-m^2x^2} dx \int_{\mathbb{R}} e^{-m^2y^2} dy = \frac{\pi}{m^2} < \infty.$$

Then K is a compact operator.

The following theorem gives conditions to ensure we can write K a a linear combination of orthogonal projections.

Theorem 2 *[without proof] Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be compact, symmetric and injective. Then*

1. *there exists a decreasing (in modulus) sequence* $\{\lambda_j\}_{j\in\mathbb{N}}$ *of eigenvalues* $|\lambda_j| \geqslant |\lambda_{j+1}|$ *with* $\lim_{j\to\infty}\lambda_j = 0$.

2. *There exists a corresponding sequence of eigenvectors* $v_j \in L_2(\mathbb{R})$ *such that* $\{v_j\}_{j\in\mathbb{N}}$ *forms an orthonormal basis of* $L_2(\mathbb{R})$.

3. *Let* $K_N = \sum_{j=0}^{N} \lambda_j P_j$, *where*

$$[P_j f](x) = v_j(x)\ (v_j, f) = \int v_j(x) v_j(y)\ f(y)$$

is the orthogonal projections on $Vect(v_j)$. *Then*

$$\lim_{N\to\infty} \|K - K^N\| = 0 \qquad \equiv \qquad K = \sum_{j=0}^{\infty} \lambda_j P_j.$$

In our case we already checked that K is compact and symmetric. It remains to verify that K is injective. We will prove the following stronger result.

Lemma 6 *Let* $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ *be defined by the kernel* $k(x, y)$ *given by* (2.3.6). *Then* $K > 0$ *as a quadratic form i.e.* $(f, Kf) > 0$ *for any function* $f \in L_2(\mathbb{R})$ *except the zero function* $f(x) = 0\ \forall x$.

Proof

$$\begin{aligned}(f, Kf) &= \int_{\mathbb{R}\times\mathbb{R}} f(x)k(x,y)f(y)dxdy \\ &= \int_{\mathbb{R}\times\mathbb{R}} g(x)e^{-\beta(x-y)^2}g(y)dxdy = \int_{\mathbb{R}} g(x)[F * g](x)dx\end{aligned}$$

where we defined

$$g(x) = f(x)e^{-\frac{m^2}{2}x^2}, \qquad F(x) = e^{-\beta x^2}.$$

The exponential factor ensures that $g \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ so the Fourier transform of g is well defined and

$$\int_{\mathbb{R}} g(x)[F * g](x)dx = \int_{\mathbb{R}} \overline{\hat{g}(k)}\ [\widehat{F * g}](k)dk = \int_{\mathbb{R}} |\hat{g}(k)|^2 \hat{F}(k)dk$$

where we used

$$\hat{g}(k) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} g(x)e^{-ikx}dx, \qquad [\widehat{F * g}](k) = \hat{F}(k)\hat{g}(k).$$

Finally

$$\hat{F}(k) = \tfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\beta x^2} e^{ikx}dx = \tfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{4\beta}} \int_{\mathbb{R}} e^{-\beta\left(x-\frac{ik}{2\beta}\right)^2}dx = e^{-\frac{x^2}{4\beta}}\tfrac{1}{\sqrt{\beta\pi}} > 0.$$

To perform the last integral we deform the contour in the complex plane and use the fact that $e^{-\beta z^2}$ is analytic, hence the integral over any closed path equals zero. Putting this results together we see that

$$(f, Kf) = \int_{\mathbb{R}} |\hat{g}(k)|^2 \hat{F}(k) dk \geqslant 0$$

Since $\hat{F}(k) > 0\ \forall k$, then $(f, Kf) = 0$ iff $\hat{g}(k) = 0\ \forall k$, iff $g(x) = 0\ \forall x$, iff $f(x) = 0$ $\forall x$. □

Consequences. Since $K > 0$ we have $0 = (f, 0) = (f, Kf) > 0$ for any $f \in \ker K$. Then $\ker K = \{0\}$, hence K is injective and the theorem above applies. Moreover, $K > 0$ implies that all eigenvalues of K must be stricly positive.

As a conclusion, in the case of our example, there exists a decreasing sequence of positive eigenvalues $\{\lambda_j\}_{j\in\mathbb{N}}$ and a correspoding sequence eigenvectors $\{v_j\}_{j\in\mathbb{N}}$ forming an orthonormal basis such that

$$\lim_{N\to\infty} \|K - K_N\| = 0 \qquad \text{where} \qquad K_N = \sum_{j=0}^{N} \lambda_j P_j.$$

As a consequence $\lim_{N\to\infty} |(u, Kw) - (u, K_N w)| = 0$ for all $u, w \in L_2(\mathbb{R})$ and (2.3.10) becomes

$$Z_\Lambda = (F^{left}, K^{2L} F^{right}) = \lim_{N\to\infty} \sum_{j=0}^{N} \lambda_j^{2L} (F^{left}, P_j F^{right})$$

2.3.3 Infinite volume limit

In the Ising case we needed two additional ingredients to control the limit as $L \to \infty$: *(a)* the largest eigenvalue is simple and *(b)* the corresponding eigenvector has strictly positive components. Since the elements of T are striclty positive Perron-Frobenius theorem ensures that both *(a)* and *(b)* are verified. Here we need a generalization of Perron-Frobenius result to integral operators.

Definition. An operator K on $L_2(\mathbb{R})$ with integral kernel $k(x, y)$ is said to have **strictly positive kernel** if for any function $f \in L_2(\mathbb{R})$ such that $f(x) \geqslant 0$ $\forall x$ and $f > 0$ on a set of positive Lebesgue measure, then $[Kf](x) > 0\ \forall x$, almost surely (i.e. except eventually on a set of measure zero). This means in particular that $k(x, y) > 0\ \forall x, y$ a.s.

Theorem 3 (Krein-Rutman) *Let K be a bounded compact symmetric operator on $L_2(\mathbb{R})$ with strictly positive kernel. Let $\lambda = \|K\|$. Then*

1. *λ is the largest eigenvalue (in absolute value) of K,*
2. *there exists an eigenvector v for λ such that $v(x) > 0\ \forall x \in \mathbb{R}$,*

3. *λ has multiplicity one.*

4. *for any eigenvalue $|\lambda'| < \lambda$, let w be an eigenvector. Then there are two sets I_1 and I_2 in $\mathbb{R}$ of positive Lebesgue measure such that $w(x) > 0\ \forall x \in I_1$ and $w(x) < 0\ \forall x \in I_2$.*

Proof Since K is compact and symmetric, then the largest eigenvalue (in absolute value) λ_0 satisfies $\|A\| = |\lambda_0| > 0$. We suppose now $\lambda_0 > 0$. We will see at the end that this must always be the case. Let v be a normalized eigenvector for λ_0. Since K is symmetric we can take v real. Then

$$\begin{aligned} 0 <&(v, Kv) = |(v, Kv)| = \left|\int v(x)k(x,y)v(y)\ dxdy\right| \\ &\leqslant \int |v(x)|\ k(x,y)\ |v(y)|\ dxdy = (|v|, K|v|). \end{aligned} \tag{2.3.11}$$

where $|v|(x) = |v(x)|$, in the first passage we used $K > 0$ (as a quadratic form) and in the last one we used $k(x,y) > 0$ (pointwise). Since v is an eigenvector for λ_0 we also have

$$\lambda_0\|v\|^2 = (v, Kv) \leqslant (|v|, K|v|) \leqslant \|K\|\ \|\ |v|\ \|^2 = \|K\|\ \|v\|^2. \tag{2.3.12}$$

But $\lambda_0 = \|K\|$ then $(v, Kv) = (|v|, K|v|)$. Now let $v(x) = v_+(x) - v_-(x)$ where

$$v_+(x) = v(x)\mathbf{1}_{v(x)>0}, \quad v_-(x) = -v(x)\mathbf{1}_{v(x)\leqslant 0}$$

hence $v_\pm(x) \geqslant 0$ for all x, $|v| = v_+ + v_-$ and

$$(v_+, Kv_-) = (v_-, Kv_+) = \int v_+(x)k(x,y)v_-(x)dx \geqslant 0$$

since all integrands are non negative. Inserting these expressions inside $(v, Kv) = (|v|, K|v|)$ we get

$$\begin{aligned} 0 < (v, Kv) &= (v_+, Kv_+) + (v_-, Kv_-) - (v_+, Kv_-) - (v_-, Kv_+) \\ &= (v_+, Kv_+) + (v_-, Kv_-) + (v_+, Kv_-) + (v_-, Kv_+) = (|v|, K|v|) \\ &\Rightarrow\ (v_+, Kv_-) + (v_-, Kv_+) = 0. \end{aligned} \tag{2.3.13}$$

Therefore

$$0 = (v_+, Kv_-) = (v_-, Kv_+) = \int v_-(x)[Kv_+](x)dx.$$

We remember that $v_+(x) \geqslant 0$ and $v_-(x) \geqslant 0$. We have two possible cases: *(a)* $v_+ > 0$ on a set of positive measure, then $[Kv_+](x) > 0\ \forall x$, then the integral above equals zero only of $v_-(x) = 0\ \forall x$, hence $v(x) = v_+(x) \geqslant 0\ \forall x$. The second possibility *(b)* is that $v_+(x) = 0\ \forall x$, then $v(x) = -v_-(x) \leqslant 0\ \forall x$. We conclude that v can be chosen to be non negative $v(x) = |v|(x) \geqslant 0 \ \forall x$. To prove strict positivity $v(x) > 0$ we observe that $\lambda_0 > 0$ then

$$v(x) = \frac{1}{\lambda_0}[Kv](x) = \frac{1}{\lambda_0}\int k(x,y)v(y)dy > 0$$

since $v(y) \geqslant 0$ and there is a set I of non zero measure such that $v(y) > 0 \ \forall y \in I$. To prove that the eigenvalue λ_0 is simple, suppose λ_0 is not simple and let v' be another eigenvector. Then we can always choose v' such that $(v, v') = 0$. Applying the arguments above to v' we conclude that $v'(x) > 0 \ \forall x$. But then

$$0 = (v, v') = \int v(x)v'(x)dx > 0,$$

that is impossible. Then λ_0 is simple. Finally, let w an eigenvector for $|\lambda'| < \lambda_0$. Since K is symmetric we must have

$$0 = (w, v_0) = \int v_0(x)w(x)dx.$$

Since $v_0(x) > 0 \ \forall x$, w must take both positive and negative values to ensure the integral is zero.

It remains to prove that $\lambda_0 > 0$. Suppose $\lambda_0 < 0$. Then repeating the same arguments as in (2.3.12) we find $-(v, Kv) = |(v, Kv)| = (|v|, K|v|)$. Then (2.3.13) becomes $(v_+, Kv_+) + (v_-, Kv_-) = 0$, hence using strict positivity of the kernel $v_+(x) = v_-(x) = 0 \ \forall x$. This ends the proof. □

Using the results above we can prove the following lemma.

Lemma 7 *Let $K : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ be defined by the kernel $k(x, y)$ given by* (2.3.6)*. Let λ_0 be the largest eigenvalue $\lambda_1 < \lambda_0$ the next eigenvalue. Let v_0 be the normalized eigenvector for λ_0 with $v_0(x) > 0 \ \forall x$ and P_0 the corresponding orthogonal projector. Then*

$$K = \lambda_0 P_0 + K_1$$

where $K_1 P_0 = P_0 K_1$ and $\|K_1\| = \lambda_1$.

Proof By Th. 3.3.11 and 3 we have

$$K_1 = \sum_{j \geqslant 1} \lambda_j P_j = K_{\mathcal{H}_1} : \mathcal{H}_1 \to \mathcal{H}_1$$

where $\mathcal{H}_1 = v_0^{\perp}$ is the subspace orthogonal to v_0 and $0 < \lambda_j \leqslant \lambda_1 < \lambda_0$ for all j. The result follows. □

2.3.4 Partition function and moments

Partition function Using the results of the previous sections we can write

$$Z_\Lambda = (F^{left}, K^{2L}F^{right}) = (F^{left}, K^{2L}F^{right}) = \lambda_0^{2L}\left[(F^{left}, P_0 F^{right}) + (F^{left}, \frac{K_1^{2L}}{\lambda_0^{2L}}F^{right})\right]$$

Since

$$|(F^{left}, \frac{K_1^{2L}}{\lambda_0^{2L}} F^{right})| \leqslant \|F^{left}\| \|F^{right}\| \left[\frac{\|K_1\|}{\lambda_0}\right]^{2L} = \|F^{left}\| \|F^{right}\| \left[\frac{\lambda_1}{\lambda_0}\right]^{2L} \to_{L\to\infty} 0$$

$$(F^{left}, P_0 F^{right}) = (F^{left}, v_0)\ (v_0, F^{right}) > 0,$$

we can write

$$\lim_{L\to\infty} \frac{\ln Z_\Lambda}{|\Lambda|} = \ln \lambda_0.$$

Magnetization Contrary the the Ising model here by symmetry we have

$$\mathbb{E}_\Lambda[\phi_j] = 0 \quad \forall j,\ \forall \Lambda.$$

To get some non trivial result we must consider ϕ^2. In the Ising case $\mathbb{E}_\Lambda[\sigma_x^2] = 1$ trivially since $\sigma^2 = 1$. On the contrary here

$$\mathbb{E}_\Lambda[\phi_j^2] = \frac{(F^{left}, K^{L+j}\Sigma^2 K^{L-j} F^{right})}{(F^{left}, K^{2L} F^{right})}$$

where we suppose $j > 0$ and we defined $[\Sigma^2 f](x) = x^2 f(x)$. Note that $[\Sigma^2 f] \notin L^2(\mathbb{R})$ in general. Let

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R}) |\ \sup_x |x|^q |f^{(p)}(x)| < \infty\ \forall q, p \geqslant 0\}$$

be the Schwartz space on $\mathbb{R}$. Then $\Sigma^2 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ Moreover $Kf \in \mathcal{S}(\mathbb{R})$ for any $f \in L^2(\mathbb{R})$ (as long as $m > 0$). Then for each finite volume Λ the expression above is finite and

$$\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j^2] = \frac{(F^{left}, P_0\Sigma^2 P_0 F^{right})}{(F^{left}, P_0 F^{right})} = (v_0, \Sigma^2 v_0) = \int x^2 v_0^2(x) dx \qquad (2.3.14)$$

Here comes a new problem: though in the discrete case the final expression was obviously finite, here the information $v_0 \in L_2(\mathbb{R})$ is not enough to garantee that the integral is finite. We will need to determine more precisely the properties of $v_0(x)$. This will be done in the next subsection.

Two point correlation Let us suppose now $(v_0, \Sigma^2 v_0) < \infty$ and consider the correlation

$$C_\Lambda^{jl} = \mathbb{E}_\Lambda[\phi_j \phi_l] = \frac{(F^{left}, K^{L+j}\Sigma K^{l-j} \Sigma K^{L-l} F^{right})}{(F^{left}, K^{2L} F^{right})}$$

where we set $0 < j < l$ and $[\Sigma f](x) = x f(x)$. As in the case of Σ^2 we have $\Sigma : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and $Kf \in \mathcal{S}(\mathbb{R})$ for any $f \in L^2(\mathbb{R})$, then

$$\begin{aligned}
\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j\phi_l] &= \frac{(F^{left}, P_0 \Sigma \frac{K^{l-j}}{\lambda_0^{l-j}} \Sigma P_0 F^{right})}{(F^{left}, P_0 F^{right})} = (v_0, \Sigma \frac{K^{l-j}}{\lambda_0^{l-j}} \Sigma v_0) \\
&= (v_0, \Sigma v_0)(v_0, \Sigma v_0) + (v_0, \Sigma \frac{K_1^{l-j}}{\lambda_0^{l-j}} \Sigma v_0) = (v_0, \Sigma \frac{K_1^{l-j}}{\lambda_0^{l-j}} \Sigma v_0) \\
&\leqslant \|v_0 \Sigma\|^2 \left(\frac{\lambda_1}{\lambda_0}\right)^{l-j} = (v_0, \Sigma^2 v_0)\ e^{-\frac{|l-j|}{\xi}}
\end{aligned}$$

where $\xi = [\ln \frac{\lambda_0}{\lambda_1}]^{-1}$.

2.3.5 Eigenvalues and eigenvectors of K

Top eigenvalue

Since the kernel $k(x, y)$ is written as product of gaussians, we can try to find an eigenvector with a gaussian form.

Lemma 8 *The function $g_\alpha(x) = e^{-\alpha x^2}$, $\alpha > 0$ is an eigenvector of K iff α*

$$\alpha = \sqrt{\beta m^2 + m^4/4}. \tag{2.3.15}$$

The corresponding eigenvalue is

$$\mu_\alpha = \sqrt{\tfrac{\pi}{\alpha+(\beta+m^2/2)}}. \tag{2.3.16}$$

Proof If we apply K to g_a we obtain

$$\begin{aligned}
[Kg_\alpha](x) &= e^{-x^2(\beta+m^2/2)} \int e^{-y^2\left[\alpha+(\beta+m^2/2)\right]} e^{+2\beta xy} dy \\
&= \sqrt{\tfrac{\pi}{\alpha+(\beta+m^2/2)}} e^{-x^2(\beta+m^2/2)} e^{\frac{4\beta^2 x^2}{4\left[\alpha+(\beta+m^2/2)\right]}} \\
&= \sqrt{\tfrac{\pi}{\alpha+(\beta+m^2/2)}} e^{-x^2\left[(\beta+m^2/2)-\frac{\beta^2}{\alpha+(\beta+m^2/2)}\right]}
\end{aligned}$$

Then $[Kg_\alpha](x) = \mu g_\alpha(x)$ iff $\mu = \mu_\alpha$ and

$$\alpha = (\beta + m^2/2) - \frac{\beta^2}{\alpha + (\beta + m^2/2)} \text{ iff } \alpha^2 = (\beta + m^2/2)^2 - \beta^2.$$

□

Note that $g_\alpha(x) > 0$ $\forall x$ then by Krein-Rutman theorem μ_α must the top eigenvalue $\mu_\alpha = \lambda_0 = \|K\|$. Let

$$v_0(x) = \left(\tfrac{2\alpha}{\pi}\right)^{\frac{1}{4}} g_\alpha(x) \tag{2.3.17}$$

be the corresponding normalized eigenvector. Then the expression $(v_0, \Sigma^2 v_0)$ in (2.3.14) is

$$(v_0, \Sigma^2 v_0) = \sqrt{\tfrac{2\alpha}{\pi}} \int x^2 e^{-2\alpha x^2} dx = \frac{1}{2\alpha} < \infty,$$

the $\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j^2]$ is finite and $\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j \phi_k] \leqslant Ce^{-|j-k|/\xi}$.

Other eigenvalues

In order to estimate the localization lenght ξ in $\mathbb{E}_\Lambda[\phi_j\phi_k]$ we need to know also the second eigenvalue λ_1. In our example, we can actually find all eigenvalues and the corresponding eigenvectors.

Lemma 9 *The eigenvalues of K are the sequence*

$$\lambda_j = \mu_\alpha \lambda_\alpha^j, j \in \mathbb{N}, \qquad \lambda_\alpha = \frac{\beta}{(\beta + m^2/2 + \alpha)}.$$

Each eigenvalue is simple and the corresponding eigenvector v_j can be written as

$$v_j(x) = (\mathrm{a}^*)^j v_0(x), \qquad \textit{where} \qquad \mathrm{a}^* = -\frac{d}{dx} + 2\alpha x,$$

and v_0 is given in (2.3.17) *above.*

Proof We remark that g_α is the solution of $g'_\alpha(x) + 2\alpha x g_\alpha(x) = 0$. Let $\mathrm{a} = \frac{d}{dx} + 2\alpha x$ and

$$\mathcal{S}(\mathbb{R}) = \{f \in C^\infty(\mathbb{R})| \sup_x |x|^q |f^{(n)}(x)| < \infty \; \forall q, p \geqslant 0\}$$

be the Schwartz space on $\mathbb{R}$. Then $\mathrm{a} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ and

$$(f, \mathrm{a}g) = (\mathrm{a}^* f, g) \qquad \forall f, g \in \mathcal{S}(\mathbb{R}).$$

Since $v_0 \in \mathcal{S}(\mathbb{R})$, $u_j = \mathrm{a}^* v_0 \in L_2(\mathbb{R})$ $\forall j > 0$. Morevover for any $f \in \mathcal{S}(\mathbb{R})$

$$\begin{aligned}
[\mathrm{a}^* K f](x) &= \left[-\frac{d}{dx} + 2\alpha x\right] \int k(x,y) f(y) dy \\
&= 2 \int \left[x\left(\beta + m^2/2 + \alpha\right) - y\beta\right] k(x,y) f(y) dy \\
[K \mathrm{a}^* f](x) &= \int k(x,y) \left[-\frac{d}{dy} + 2\alpha y\right] f(y) dy \\
&= 2 \int \left[x\beta - y\left(\beta + m^2/2 - \alpha\right)\right] k(x,y) f(y) dy \\
&= \frac{\beta}{(\beta + m^2/2 + \alpha)} [\mathrm{a}^* K f](x)
\end{aligned}$$

where we used $\alpha^2 = (\beta + m^2/2)^2 - \beta^2$. Taking $f = v_0$ we obtain immediately that v_j is a sequence of eigenvectors for the eigenvalues λ_j. Since $\lambda_j \neq \lambda_k$ $\forall j \neq k$ and $K^* = K$ the eigenvectors are orthogonal

$$\lambda_j (v_j, v_k) = (K v_j, v_k) = (v_j, K v_k) = \lambda_k (v_j, v_k).$$

More precisely, using $[a, a^*] = 4\alpha \mathrm{Id}$ and

$$[a, (a^*)^k] = [a, a^*](a^*)^{k-1} + a^*[a, (a^*)^{k-1}] = 4\alpha (a^*)^{k-1} + a^*[a, (a^*)^{k-1}] = 4\alpha k (a^*)^{k-1}$$

we obtain

$$(v_j, v_k) = \delta_{jk} \frac{(4\alpha)^j}{j!}.$$

Finally we remark that $v_j(x) = p_j(x) e^{-\alpha x^2}$ where $p_j(x)$ is a polynomial of order j and

$$\int e^{-2\alpha x^2} p_j(x) p_k(x) = c_j \delta_{jk},$$

where c_j is some positive constant. Precisely

$$\|g_\alpha\| \; p_j(x) = e^{\alpha x^2} (a^*)^j e^{-\alpha x^2} = e^{2\alpha x^2} \left(-\tfrac{d}{dx}\right)^j e^{-2\alpha x^2} = (2\alpha)^{j/2} H_j(x\sqrt{2\alpha})$$

where we used

$$\left(-\tfrac{d}{dx} + 2\alpha x\right) e^{\alpha x^2} = e^{\alpha x^2} \left(-\tfrac{d}{dx}\right)$$

and

$$H_j(x) = e^{+x^2} \left(-\tfrac{d}{dx}\right)^j e^{-x^2} = e^{+\frac{x^2}{2}} \left(-\tfrac{d}{dx} + x\right)^j e^{-\frac{x^2}{2}}$$

is the Hermite polynomial of order j. Since Hermite polynomials span $L_2(\mathbb{R})$, by Th. 3.3.11 above the family $\{v_j\}_{j \in \mathbb{N}}$ contains all eigenvectors. □

2.4 Conclusions, remarks

In this chapter we have considered the one dimensional version of two models: the Ising model and the harmonic cristal. In both cases we have applied the transfer matrix approach to study the infinite volume limit. Below is a summary of the results we obtained.

2.4.1 Hamiltonians

The starting hamiltonians for the Ising (resp. harmonic cristal) model are

$$H^I_\Lambda(\sigma) = -\sum_{j=-L}^{L-1} \sigma_j \sigma_{j+1} - \frac{h}{\beta} \sum_{j=-L}^{L} \sigma_j, \qquad \sigma \in \Omega_\Lambda = \{1, -1\}^\Lambda$$

$$H^{har}_\Lambda(\phi) = \sum_{j=-L}^{L-1} (\phi_j - \phi_{j+1})^2 + m^2 \sum_{j=-L}^{L} \phi_j^2, \qquad \phi \in \Omega_\Lambda = \mathbb{R}^\Lambda$$

Boundary conditions. In the Ising case we have considered three types of boudary conditions:

$$\bar\sigma: \quad H^{\bar\sigma}_\Lambda(\sigma) = H^I(\sigma) - J\left(\sigma_{-L}\bar\sigma_{-L-1} + \sigma_L \bar\sigma_{L+1}\right)$$

$$\text{periodic:} \quad H^{per}_\Lambda(\sigma) = H^I(\sigma) - J\sigma_L \sigma_{-L}$$

$$\text{free:} \quad H^{free}_\Lambda(\sigma) = H^I(\sigma).$$

The corresponding boundary conditions in the case of the harmonic cristal are

$$\begin{array}{lll} \text{Dirichlet:} & H_\Lambda^D(\phi) = H_\Lambda^{har}(\phi) + \phi_{-L}^2 + \phi_L^2 & \to \phi_{L+1} = \phi_{-L-1} = 0 \\ \text{periodic:} & H_\Lambda^{per}(\phi) = H_\Lambda^{har}(\phi) + (\phi_L - \phi_{-L})^2 & \\ \text{Neuman:} & H_\Lambda^N(\phi) = H^{har}(\phi) & \to [\nabla\phi]_{\partial\Lambda} = 0. \end{array}$$

2.4.2 Partition function

In both models we wrote the partition function in terms of a transfer operator. As a result

$$\lim_{L\to\infty} \frac{\ln Z_\Lambda^I}{|\Lambda|} = \ln\lambda_1 = \ln[e^\beta \cosh h + \sqrt{(e^\beta \sinh h)^2 + e^{-2\beta}}]$$
$$\lim_{L\to\infty} \tfrac{\ln Z_\Lambda^{har}}{|\Lambda|} = \ln\lambda_0 = \tfrac{1}{2}\ln\tfrac{\pi}{\alpha+(\beta+m^2/2)}, \quad \alpha = \sqrt{m^2\beta + \tfrac{m^4}{4}},$$

where λ_1 (resp. λ_0) is the largest eigenvalue of the transfer matrix T (resp. the transfer operator K). These limits are independent from the boundary conditions.

2.4.3 Magnetization

For the magnetization we obtained

$$\lim_{L\to\infty} \mathbb{E}_\Lambda[\sigma_j] = (v_1, \Sigma v_1) = M(h) \to \begin{cases} \pm 1 & h \to \pm\infty \\ 0 & h \to 0 \end{cases}$$
$$\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j] = 0$$
$$\lim_{L\to\infty} \mathbb{E}_\Lambda[\sigma_j^2] = 1$$
$$\lim_{L\to\infty} \mathbb{E}_\Lambda[\phi_j^2] = (v_0, \Sigma^2 v_0) = \tfrac{1}{4\alpha} \to \begin{cases} 0 & m \to \infty \\ +\infty & m \to 0 \end{cases}$$

In both cases the result is independent from the position j and from the boundary conditions. Note that though the averages spin is always finite, the average ϕ^2 diverges as $m \to 0$, reflecting the fact that ϕ_j is an unbounded variable and the fluctuations become very large when m is small.

2.4.4 Correlations

We have considered only two point correlations functions:

$$\lim_{L\to\infty} (\mathbb{E}_\Lambda[\sigma_i\sigma_j] - \mathbb{E}_\Lambda[\sigma_i]\mathbb{E}_\Lambda[\sigma_j]) = Ce^{-\frac{|i-j|}{\xi}}, \xi = \frac{1}{\ln\frac{\lambda_1}{\lambda_2}} \to \begin{cases} 0 & h \to \pm\infty \\ \frac{1}{\ln\frac{\cosh\beta}{\sinh\beta}} & h \to 0 \end{cases}$$
$$\lim_{L\to\infty} (\mathbb{E}_\Lambda[\phi_i\phi_j] - \mathbb{E}_\Lambda[\phi_i]\mathbb{E}_\Lambda[\phi_j]) \leqslant \tfrac{1}{4\alpha} e^{-\frac{|i-j|}{\xi}}, \xi = \frac{1}{\ln\frac{\beta+m^2/2+\alpha}{\beta}} \to \begin{cases} 0 & m \to \infty \\ +\infty & m \to 0 \end{cases}$$

Note that the correlation length ξ is always finite in the Ising model (unless $\beta \to \infty$). On the contrary, ξ diverges as $m \to 0$ in the harmonic cristal. Since the prefactor $1/\alpha$ also diverges it is better to consider the expression

$$\lim_{L\to\infty} \frac{(\mathbb{E}_\Lambda[\phi_i\phi_j] - \mathbb{E}_\Lambda[\phi_i]\mathbb{E}_\Lambda[\phi_j])}{\sqrt{\mathbb{E}_\Lambda[\phi_i^2]\mathbb{E}_\Lambda[\phi_j^2]}} \leqslant e^{-\frac{|i-j|}{\xi}}.$$

It is important to remark that the divergent quantities in the harmonic cristal appear *for any choice* of the boundary conditions.

2.4.5 Generalizations

The transfer matrix approach may be applied to much more general situations. One may for example replace the quadratic potential $m^2\phi^2$ by some function $V(\phi)$ such that

- $V(\phi) \to \infty$ as $|\phi| \to \infty$
- $V(0) = 0$ and V has a unique minimum at $\phi = 0$.

These conditions garantee that $V(\phi) = m^2\phi^2 + O(\phi^3)$ near $\phi = 0$. Then when β is large the transfer matrix is well approximated (see [**?**, Ch. 5] for more details) by the harmonic transfer matrix we already studied. Some examples of such potential are $V(\phi) = \phi^4$ or $V(\phi) = \ln(1 + \phi^2)$. Note that in the second example we cannot study high order correlation functions since $\mathbb{E}_\Lambda[\phi_i^n]$ since the log-potential does not garantee that the integral remains finite. More work is needed when the potential $V(\phi)$ has several minima.

When the transfer matrix is real but not symmetrix, or complex but not self-adjoint, then most of the theorems we used do not apply! Situations when one can still do something are

- the transfer operator K is real with (non strictly) positive kernel (not necessarily symmetric) such that *some power of* K has strictly positive kernel.
- the transfer operator is complex and normal.

Chapter 3

Higher dimensional problems

In one dimension, the transfer matrix approach garantees the existence of the infinite volume limit, as long as the transfer operator is regular enough. When in addition we can show that this operator is "near" to the harmonic cristal, then we can obtain precise estimates of the limit.

In dimension larger than one, the transfer matrix approach does not apply but as in the 1d case, many techniques use some kind of comparison with the harmonic cristal. We will then start the chapter reviewing the results we obtained for the harmonic cristal in $d = 1$ with a different approach that, contrary to the transfer matrix, can be directly generalized to any dimension.

3.1 Gaussian integrals in 1d

3.1.1 The harmonic cristal as a gaussian integral

The Hamiltonian for the harmonic cristal we introduced in the previous chapter can be written as a quadratic form

$$\beta H_\Lambda^{(har)}(\phi) = (\phi, A_\Lambda^{(har)}\phi)_\Lambda = \sum_{j,k=-L}^{L} \phi_j A_{\Lambda\ jk}^{(har)} \phi_k = \sum_{j=-L}^{L-1} \beta(\phi_j - \phi_{j+1})^2 + \sum_{j=-L}^{L} m^2 \phi_j^2$$
$$= (\phi, -\beta\Delta_\Lambda \phi) + (\phi, m^2 \mathrm{I}_\Lambda \phi),$$

where $(\phi, \psi)_\Lambda = \sum_{j=-L}^{L} \phi_j \psi_j$ is the real euclidean scalar product on Λ and $-\Delta_\Lambda$ is the discrete Laplacian defined by

$$(-\Delta_\Lambda)_{ij} = \begin{cases} -1 & |i-j| = 1 \\ \sum_{k\in\Lambda,\ |k-j|=1} 1 & i = j \end{cases}$$

Inserting the boundary conditions the Hamiltonian becomes

$$\begin{array}{ll} \text{Dirichlet:} & H^D_\Lambda(\phi) = H^{har}_\Lambda(\phi) + \phi^2_{-L} + \phi^2_L = (\phi, [-\beta\Delta^D_\Lambda + m^2 \mathrm{I}_\Lambda]\phi) \\ \text{periodic:} & H^{per}_\Lambda(\phi) = H^{har}_\Lambda(\phi) + (\phi_L - \phi_{-L})^2 = (\phi, [-\beta\Delta^{per}_\Lambda + m^2 \mathrm{I}_\Lambda]\phi) \\ \text{Neuman:} & H^N_\Lambda(\phi) = H^{har}(\phi) = (\phi, [-\beta\Delta^N_\Lambda + m^2 \mathrm{I}_\Lambda]\phi) \end{array}$$

where

$$-\Delta^N = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad -\Delta^D = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$-\Delta^{per} = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{pmatrix}$$

Note that

$$0 \leqslant (f, -\Delta^N f) \leqslant (f, -\Delta^{per} f) \leqslant 2(f, -\Delta^D f)$$

where in the last inequality we used $2(f^2_{-L} + f^2_L) \geqslant (f_{-L} - f_{-L})^2$. Moreover the constant vector is in the kernel of both Δ^N and Δ^{per}

$$-\Delta^N f = -\Delta^{per} f = 0 \quad \text{if} \quad f_j = f \ \forall j,$$

while $(f, -\Delta^D f) > 0 \ \forall f \in \mathbb{R}^\Lambda$. Therefore only the meaure $d\mu^D_\Lambda(\phi)$ with Dirichlet boundary conditions is well defined also for $m = 0$.

3.1.2 Gaussian integrals and correlations

In the following we will need some basic facts about gaussian meaures.

Lemma 10 *Let A be a $N \times N$ real symmetric matrix such that $A > 0$ as a quadratic form. Let $d\phi = \prod_{j=1}^N d\phi_j$ the Lebesgue measure on $\mathbb{R}^N$. Then*

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} d\phi = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}, \qquad \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} \phi_{j_1}\phi_{j_2} d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} d\phi} = A^{-1}_{ij}.$$

More generally let $j_1, \ldots, j_n \in \{1, \ldots, N\}$ n (not necessarily different) points. Then

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} \phi_{j_1}\phi_{j_2} \cdots \phi_{j_n} d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi, A\phi)} d\phi} = \begin{cases} 0 & n \text{ odd} \\ \sum_P \prod_{(\alpha,\beta)\in P} A^{-1}_{j_\alpha j_\beta} & n = 2m \end{cases}$$

where P is a pairing of the set the set $\{1, \ldots, 2m\}$, i.e. a partition into m subsets of size 2.

Example

Proof Since A is real and symmetric, there exist an real orthogonal matrix U ($U^t = U^{-1}$) and a real diagonal matrix $\hat\lambda$ such that $A = U^t \hat\lambda U$ and

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} d\phi = \int_{\mathbb{R}^N} e^{-\frac{1}{2}(U\phi,\hat\lambda U\phi)} d\phi = \int_{\mathbb{R}^N} e^{-\frac{1}{2}(\tilde\phi,\hat\lambda\tilde\phi)} |\det U^{-1}| d\tilde\phi$$
$$= \prod_{i=1}^N \int_{\mathbb{R}} e^{-\frac{1}{2}\lambda_i \tilde\phi_j^2} d\tilde\phi_i = \frac{(2\pi)^{N/2}}{\prod_{i=1}^N \sqrt{\lambda_i}} = \frac{(2\pi)^{N/2}}{\sqrt{\det A}}.$$

where we performed the change of variable $\tilde\phi = U\phi$ and we used $|\det U| = 1$. To prove the other relation we may use integration by parts. We have

$$\phi_{j_1} e^{-\frac{1}{2}(\phi,A\phi)} = -\sum_{i=1}^N A^{-1}_{j_1 i} \, \frac{\partial}{\partial\phi_i} e^{-\frac{1}{2}(\phi,A\phi)}.$$

Inserting this relation in the integral we obtain

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} \phi_{j_1}\phi_{j_2} d\phi = -\sum_{i=1}^N A^{-1}_{j_1 i} \int_{\mathbb{R}^N} \phi_{j_2} \frac{\partial}{\partial\phi_i} e^{-\frac{1}{2}(\phi,A\phi)} d\phi$$
$$= +\sum_{i=1}^N A^{-1}_{j_1 i} \int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} \frac{\partial}{\partial\phi_i}\phi_{j_2} d\phi = A^{-1}_{j_1 j_2} \int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} d\phi$$

The proof for the general case is similar. Alternatively one may use the *generating function* $S : \{f_j\}_{j=1}^N \to \mathbb{R}$

$$S(f) = \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} e^{(\phi,f)} d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} d\phi}$$
$$= e^{\frac{1}{2}(f,A^{-1}f)} \frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}([\phi-A^{-1}f],A[\phi-A^{-1}f])} d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} d\phi} = e^{\frac{1}{2}(f,A^{-1}f)}$$

Since S is smooth in f_j $\forall j$ we have

$$\frac{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} \phi_{j_1}\phi_{j_2}\cdots\phi_{j_n} d\phi}{\int_{\mathbb{R}^N} e^{-\frac{1}{2}(\phi,A\phi)} d\phi} = \frac{\partial^n}{\partial f_{j_1}\cdots\partial f_{j_n}} S(f)_{|f=0}.$$

□

3.1.3 Partition function and correlations

With these formulas we can now compute the partition function and correlation functions for the harmonic cristal in $d = 1$

$$Z_\Lambda^{(b.c.)} = \int e^{-\beta H_\Lambda^{(b.c.)}(\phi)} d\phi = \int e^{-(\phi,A_\Lambda\phi)_\Lambda} d\phi = \frac{(\pi)^{\frac{2L+1}{2}}}{\sqrt{\det A_\Lambda}}$$
$$\mathbb{E}_\Lambda^{(b.c.)}[\phi_x^2] = \frac{\int e^{-(\phi,A_\Lambda\phi)_\Lambda} \; \phi_x^2 \; d\phi}{\int e^{-(\phi,A_\Lambda\phi)_\Lambda} \; d\phi} = \frac{1}{2}(A_\Lambda^{-1})_{xx}$$
$$\mathbb{E}_\Lambda^{(b.c.)}[\phi_x\phi_y] = \frac{\int e^{-(\phi,A_\Lambda\phi)_\Lambda} \; \phi_x\phi_y \; d\phi}{\int e^{-(\phi,A_\Lambda\phi)_\Lambda} \; d\phi} = \frac{1}{2}(A_\Lambda^{-1})_{xy}.$$

where the matrix A_Λ depends on the boundary conditions. The problem is then converted in the study of the determinant and inverse of A_Λ as $\Lambda \to \mathbb{Z}$.

3.1.4 Finite volume computation: periodic boundary conditions

In the case of periodic boundary conditions we can compute the eigenvalues and eigenvectors of the discrete Laplacian by taking the Fourier transform.

Discrete Fourier transform

Any function $f \in \mathbb{R}^\Lambda$ can be seen as a periodic function of period $T = 2L + 1$, i.e. $f \in \mathbb{R}^\mathbb{Z}$ with $f(x + nT) = f(x)$ $\forall n \in \mathbb{Z}$. Let $\mathcal{P}_T(\mathbb{Z})$ the corresponding set of functions.

Definition 1 (Discrete Fourier transform) *The discrete Fourier transform is a linear functional*

$$\begin{aligned} \mathcal{F}: \quad \mathcal{P}_T(\mathbb{Z}) &\to \mathcal{P}_T(\mathbb{Z}) \\ f &\to \mathcal{F}[f](n) = \hat{f}(n) = c_1 \sum_{x=-L}^{L} f(x) e^{-ik_n x} \end{aligned}$$

where $n \in \Lambda = \{-L, \ldots, L\}$, $k_n = \frac{2\pi n}{2L+1}$ *and* $c_1 > 0$ *is a normalization constant. This functional is invertible and*

$$\begin{aligned} \mathcal{F}^{-1}: \quad \mathcal{P}_T(\mathbb{Z}) &\to \mathcal{P}_T(\mathbb{Z}) \\ g &\to \mathcal{F}^{-1}[g](x) = \check{g}(x) = c_2 \sum_{n=-L}^{L} g(k_n) e^{+ik_n x} \end{aligned}$$

where the constants $c_1, c_2 > 0$ *must satisfy* $c_1 c_2 = \frac{1}{2L+1}$.

There are several possible conventions. One may take $c_1 = c_2 = (2L + 1)^{-1/2}$, or $c_1 = 1$ and $c_2 = (2L + 1)^{-1}$.

With these definitions we have the following properties

Convolution. Let $f, g \in \mathcal{P}_T(\mathbb{Z})$. The (discrete) convolution is defined by

$$f * g(x) = \sum_{y=-L}^{L} f(x - y) g(y)$$

The corresponding Fourier transform is

$$[\mathcal{F}(f * g)](k_n) = c_1 c_2^2 (2L + 1)^2 \hat{f}(k_n) \hat{g}(k_n) = \frac{1}{c_1} \hat{f}(k_n) \hat{g}(k_n).$$

Then

$$\mathcal{F}^{-1}[f \cdot g](x) = c_1 (\check{f} * \check{g})(x).$$

Scalar product. Let $f, g \in \mathcal{P}_T(\mathbb{Z})$. We consider the real scalar product on Λ $(f, g) = \sum_{x=-L}^{L} f(x)g(x)$. Then we have

$$(f, g) = c_2^2(2L+1) \sum_{n=-L}^{L} \overline{\hat{f}(k_n)}\hat{g}(k_n) = \frac{1}{c_1^2}\frac{1}{2L+1} \sum_{n=-L}^{L} \overline{\hat{f}(k_n)}\hat{g}(k_n)$$

Fourier transform of the Laplacian. Note that the matrix $-\Delta_\Lambda^{per}$ is translation invariant i.e.

$$(-\Delta_\Lambda^{per})_{x,y} = (-\Delta_\Lambda^{per})_{x-y,0} F(|x-y|)$$

since the value of this matrix element depends only on the distance $|x-y|$, then it acts as a convolution

$$[(-\Delta_\Lambda^{per})f](x) = \sum_y (-\Delta_\Lambda^{per})_{x,y} f(y) = [F * f](y).$$

The Fourier transform is then

$$\mathcal{F}[(-\Delta_\Lambda^{per})f](k_n) = [\mathcal{F}(F * f)](k_n) = \frac{1}{c_1}\hat{F}(k_n)\hat{f}(k_n).$$

Therefore, by translation invariance the Laplacian is a diagonal matrix in Fourier space and the eigenvalues are given by

$$\lambda_n = \frac{1}{c_1}\hat{F}(k_n).$$

To compute the eigenvalues

$$\frac{1}{c_1}\hat{F}(k_n) = \sum_{x=-L}^{L} e^{-ik_n x}(-\Delta_\Lambda^{per})_{x,0} = \left[2 - e^{-ik_n} - e^{ik_n}\right] = 2[1-\cos(k_n)]$$

Note that by symmetry there are $L+1$ distinct eigenvalues: $\lambda_n = 2[1-\cos(k_n)]$ with $n = 1, .., L$ each of multiplicity 2 and $\lambda_0 = 0$ of multiplicity 1.

Let $M = -\beta\Delta_\Lambda^{per} + m^2 \mathrm{I}_\Lambda$. From above we have

$$\widehat{[Mf]}(k_n) = \sum_m \hat{M}_{k_n k_m}\hat{f}(k_m) = \mu(k_n)\hat{f}(k_n) = [\mu\cdot\hat{f}](k_n) \qquad \text{where } \mu(k_n) = 2\beta(1-\cos k_n)+m^2.$$

Hence $\hat{M}_{k_n k_m} = \delta_{nm}\mu(k_n)/c_1$ is a diagonal matrix and

$$[\hat{M}^{-1}\hat{f}](k_n) = [\mu^{-1}\cdot\hat{f}](k_n) = \frac{1}{\mu(k_n)}\hat{f}(k_n).$$

Therefore

$$[M^{-1}f](x) = \sum_y M_{xy}^{-1} f(y) = \mathcal{F}^{-1}[\hat{M}^{-1}\hat{f}](x) = \mathcal{F}^{-1}[\mu^{-1}\cdot\hat{f}](x)$$
$$= c_1\left[\mathcal{F}^{-1}(\mu^{-1}) * f\right](x) = c_1 \sum_y \mathcal{F}^{-1}(\mu^{-1})(x-y)f(y)$$

As a conclusion we obtain

$$M_{xy}^{-1} = c_1 \mathcal{F}^{-1}(\mu^{-1})(x-y) = c_1 c_2 \sum_{n=-L}^{L} \tfrac{1}{\mu(k_n)} e^{ik_n(x-y)}$$
$$= \tfrac{1}{2L+1} \sum_{n=-L}^{L} \tfrac{1}{\mu(k_n)} e^{ik_n(x-y)} = \tfrac{1}{2L+1} \sum_{n=-L}^{L} \tfrac{e^{ik_n(x-y)}}{2\beta(1-\cos k_n)+m^2}.$$

This result is independent from the choice of c_1, c_2.

Remark. The arguments above are false if we take Dirichlet or Neuman boundary conditions.

Finite volume partition function and correlations

With the definitions above we can now explicitely compute some quantities. Since we are considering periodic boundary conditions we have $A_\Lambda = M$. Moreover each eigenvalue except $\mu(0)$ has multiplicity 2, then

$$\det M = \mu(0) \prod_{n=1}^{L} \mu(k_n)^2 = m^2 \prod_{n=1}^{L} \mu(k_n)^2.$$

Then

$$\tfrac{1}{2L+1} \ln Z_\Lambda^{(per)} = \ln \sqrt{\pi} - \tfrac{\ln m}{2L+1} - \tfrac{1}{2L+1} \sum_{n=1}^{L} \ln \mu(k_n)$$
$$\mathbb{E}_\Lambda^{per}[\phi_x^2] = \tfrac{1}{2}(M^{-1})_{xx} = \tfrac{1}{2m^2|\Lambda|} + \tfrac{1}{|\Lambda|} \sum_{n=1}^{L} \frac{1}{\mu(k_n)}$$
$$\mathbb{E}_\Lambda^{per}[\phi_x \phi_y] = \tfrac{1}{2}(M^{-1})_{xy} = \tfrac{1}{2m^2|\Lambda|} + \tfrac{1}{|\Lambda|} \sum_{n=1}^{L} \frac{\cos(k_n(x-y))}{\mu(k_n)}.$$

where $|\Lambda| = 2L+1$ and we used $k_{-n} = -k_n$ and $\mu(k_n) = \mu(-k_n)$.

Some elementary estimates on the two point function: spectral gap
Contrary to the continuous Laplacian, the discrete Laplacian has a spectral gap,

$$\mu(k_n) - \mu(k_0) \geqslant 2\beta(1-\cos(\tfrac{2\pi}{2L+1})) = O(L^{-2}) > 0 \qquad \forall n \neq 0.$$

Using this fact we can prove the following estimates.

Lemma 11 *There exist constants C_1, C_2 such that*

$$|\mathbb{E}_\Lambda^{per}[\phi_x \phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x \phi_x]| \leqslant \frac{C_1}{m} \tag{3.1.1}$$
$$|\mathbb{E}_\Lambda^{per}[\phi_x \phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x \phi_{y+1}]| \leqslant C_2 \tag{3.1.2}$$

for any choice of m, $|\Lambda|$, x and y. The factor m^{-1} is due to the properties of the one dimensional Laplacian and cannot be avoided. Precisely, there exist two constants K_1, K_2 such that

$$\frac{K_1}{m} \leqslant \left| \mathbb{E}_{\Lambda}^{per}[\phi_x \phi_x] - \tfrac{1}{2m^2|\Lambda|} \right| \leqslant \frac{K_2}{m} \tag{3.1.3}$$

for any choice of m and $|\Lambda|$. The points on the boundary play a special role and the corresponding two point function has nicer a priori bounds. Precisely there exists a constant C_3 such that

$$m \left| \mathbb{E}_{\Lambda}^{per}[\phi_x \phi_{\pm L}] - \tfrac{1}{2m^2|\Lambda|} \right| \leqslant \frac{C_3}{mL} \tag{3.1.4}$$

for any choice of m Λ and x.

Proof.

$$|\mathbb{E}_{\Lambda}^{per}[\phi_x \phi_y] - \mathbb{E}_{\Lambda}^{per}[\phi_x \phi_x]| = \left| \tfrac{1}{|\Lambda|} \sum_{n=1}^{L} \tfrac{[\cos(k_n(x-y))-1]}{\mu(k_n)} \right|$$
$$\leqslant \tfrac{2}{|\Lambda|} \sum_{n=1}^{L} \tfrac{1}{2\beta(1-\cos k_n)+m^2}$$
$$\leqslant \tfrac{2}{|\Lambda|} \sum_{1 \leqslant n \leqslant L/10} \tfrac{1}{2\beta(1-\cos k_n)+m^2} + \tfrac{2}{|\Lambda|} \sum_{L/10 < n \leqslant L} \tfrac{1}{2\beta(1-\cos k_n)+m^2}$$

To estimate the second sum notice that

$$1 - \cos(k_n) \geqslant 1 - \cos \pi/10 + O(L^{-1}) \geqslant Const \qquad \forall \quad L/10 < n \leqslant L.$$

Then

$$\tfrac{2}{|\Lambda|} \sum_{L/10 < n \leqslant L} \tfrac{1}{2\beta(1-\cos k_n)+m^2} \leqslant \tfrac{2L}{2L+1} \sup_{L/10 < n \leqslant L} \tfrac{1}{2\beta(1-\cos k_n)+m^2} \leqslant Const.$$

To estimate the first sum notice that we can find a small number $\rho > 0$ such that

$$1 - \cos(k_n) \geqslant \rho k_n^2 \qquad \forall \quad n \leqslant L/10.$$

Then

$$\tfrac{2}{|\Lambda|} \sum_{1 \leqslant n \leqslant L/10} \tfrac{1}{2\beta(1-\cos k_n)+m^2} \leqslant \tfrac{1}{\pi} \tfrac{2\pi}{|\Lambda|} \sum_{1 \leqslant n \leqslant L/10} \tfrac{1}{2\beta\rho k_n^2+m^2}$$
$$\leqslant \tfrac{1}{\pi} \int_{a_L}^{b_L} \tfrac{1}{2\beta\rho k^2+m^2} dk = \tfrac{1}{m\pi\sqrt{2\beta\rho}} \int_{\frac{a_L\sqrt{2\beta\rho}}{m}}^{\frac{b_L\sqrt{2\beta\rho}}{m}} \tfrac{1}{k^2+1} dk$$
$$= \tfrac{1}{m\pi\sqrt{2\beta\rho}} [\arctan(k)]_{\frac{a_L\sqrt{2\beta\rho}}{m}}^{\frac{b_L\sqrt{2\beta\rho}}{m}} \leqslant \tfrac{Const}{m}$$

where we set $a_L = \frac{2\pi}{2L+1}$, $b_L = \frac{\pi + O(L^{-1})}{10}$. The estimate (3.1.3) is proved in the same way. To obtain the lower bound notice that $1 - \cos(k_n) \leqslant \rho k_n^2$ for some constant ρ for all $1 \leqslant n \leqslant L/10$ and $1 - \cos(k_n) \leqslant 2$ for all $0 \leqslant n \leqslant L$. To prove (3.1.2)

$$\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x\phi_{y+1}] = \frac{1}{|\Lambda|}\sum_{n=1}^{L} \frac{[\cos(k_n(x-y)) - \cos(k_n(x-y-1))]}{\mu(k_n)}$$

$$= \frac{1}{|\Lambda|}\sum_{n=1}^{L} \frac{\cos(k_n(x-y))[1-\cos(k_n)] - \sin(k_n(x-y))\sin(k_n)}{\mu(k_n)}$$

Then

$$|\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] - \mathbb{E}_\Lambda^{per}[\phi_x\phi_{y+1}]| \leqslant \frac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{[1-\cos(k_n)]}{2\beta(1-\cos(k_n))+m^2} + \frac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{|\sin(k_n(x-y))\sin(k_n)|}{2\beta(1-\cos(k_n))+m^2}$$

$$\leqslant \frac{L}{2\beta(2L+1)} + \frac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{|\sin(k_n(x-y))\sin(k_n)|}{2\beta(1-\cos(k_n))+m^2}$$

To estimate the last term we break it as before in two sums $\sum_{n\leqslant\bar{n}}$ and $\sum_{\bar{n}\leqslant n\leqslant L}$, where

$$\bar{n} = \frac{L}{10|x-y|} \quad ,1 \leqslant |x-y| \ll L \Rightarrow \quad |k_n| \leqslant \frac{L}{10},\ \text{and}\ |k_n(x-y)| \leqslant \frac{L}{10}\ \forall\ n \leqslant \bar{n}.$$

The last sum is bounded by a constant. The first sum is bounded by

$$\frac{K}{|\Lambda|}\sum_{1\leqslant n\leqslant\bar{n}}\frac{k_n^2|x-y|}{2\beta\rho k_n^2+m^2} \leqslant K\frac{|x-y|\bar{n}}{2\beta\rho|\Lambda|} \leqslant K$$

where K and K are some constants. Finally to prove (3.1.4) note that

$$\left|\mathbb{E}_\Lambda^{per}[\phi_x\phi_L] - \frac{1}{2m^2|\Lambda|}\right| = \frac{1}{|\Lambda|}\left|\sum_{n=1}^{L}\frac{\cos k_n(x-L)}{2\beta(1-\cos(k_n))+m^2}\right| = \frac{1}{|\Lambda|}\left|\sum_{n=1}^{L}\frac{\cos[k_n(x-1/2)+n\pi]}{2\beta(1-\cos(k_n))+m^2}\right|$$

$$= \frac{1}{|\Lambda|}\left|\sum_{n=1}^{L}(-1)^n f(k_n)\right| = \frac{1}{|\Lambda|}\left|\sum_{1\leqslant n\leqslant L/2} f(k_{2n}) - f(k_{2n-1})\right| \leqslant \frac{1}{|\Lambda|}\sum_{n=1}^{L}|f(k_{n+1}) - f(k_n)|.$$

where

$$f(k) = \frac{\cos[k(x-1/2)]}{2\beta(1-\cos(k))+m^2}.$$

Now

$$f(k_{n+1}) - f(k_n) = f'(k^*)\delta k, \qquad k_n \leqslant k^* \leqslant k_{n+1}, \quad \delta k = \frac{2\pi}{2L+1}.$$

There exist constants $C_1(x,\beta), C_2(x,\beta)$ such that

$$|f'(k)| \leqslant \frac{|\sin(k(x-1/2))|}{[2\beta(1-\cos(k))+m^2]} + \frac{|\cos(k(x-1/2))||\sin(k)|}{[2\beta(1-\cos(k))+m^2]^2}$$

$$\leqslant \begin{cases} C_1(x,\beta) & \forall \frac{\pi}{10|x|} \leqslant k \leqslant \pi \\ C_2(x,\beta)\left[\frac{k}{2\beta\rho k^2+m^2} + \frac{k}{[2\beta\rho k^2+m^2]^2}\right] & \forall 0 \leqslant k \leqslant \frac{\pi}{10|x|} \end{cases}$$

Inserting these bounds in the sum above we obtain

$$\tfrac{1}{|\Lambda|}\sum_{n=1}^{L}|f(k_{n+1})-f(k_n)| \leqslant C_1(x,\beta)\delta k + C_2(x,\beta)\delta k\int_0^{\pi/10|x|}\left[\tfrac{k}{2\beta\rho k^2+m^2}+\tfrac{k}{[2\beta\rho k^2+m^2]^2}\right]dk$$
$$= O(L^{-1})+O(L^{-1})|\ln m| + O\left(\tfrac{1}{m^2L}\right) = \tfrac{1}{m}O\left(\tfrac{1}{mL}\right).$$

This proves the result.

■

3.1.5 Infinite volume limit for periodic boundary conditions

When $L\to\infty$ the Riemann sums become integrals

$$\begin{aligned}
\tfrac{1}{2L+1}\ln Z_\Lambda^{(per)} &= \ln\sqrt{\pi} - \tfrac{\ln m}{2L+1} - \tfrac{1}{2L+1}\sum_{n=1}^{L}\ln\mu(k_n)\\
&\to_{L\to\infty} \ln\sqrt{\pi} - \tfrac{1}{2\pi}\int_0^\pi \ln[2\beta(1-\cos k)+m^2]dk.\\
\mathbb{E}_\Lambda^{per}[\phi_x^2] &= \tfrac{1}{2m|\Lambda|} + \tfrac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{1}{\mu(k_n)} \to_{L\to\infty} \tfrac{1}{2\pi}\int_0^\pi \frac{1}{2\beta(1-\cos k)+m^2}dk\\
\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] &= \tfrac{1}{2m|\Lambda|} + \tfrac{1}{|\Lambda|}\sum_{n=1}^{L}\frac{\cos(k_n(x-y))}{\mu(k_n)}\\
&\to_{L\to\infty} \tfrac{1}{2\pi}\int_0^\pi \frac{\cos(k(x-y))}{2\beta(1-\cos k)+m^2}dk = \tfrac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{ik(x-y)}}{2\beta(1-\cos k)+m^2}dk
\end{aligned}$$

Lemma 12 *The limits obtained above coincide with the results we obtained by transfer matrix approach. In particular the two-point correlations are given by*

$$\begin{aligned}
\lim_{L\to\infty}\mathbb{E}_\Lambda^{per}[\phi_x^2] &= \frac{1}{4\alpha}\\
\lim_{L\to\infty}\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] &= \frac{1}{4\alpha}z_1^{|x|}
\end{aligned}$$

where

$$z_1 = \left(1+\tfrac{m^2}{2\beta}\right) - \sqrt{\left(1+\tfrac{m^2}{2\beta}\right)^2 - 1}.$$

Proof. The two point function is symmetric under exchange of x and y so we can always choose $x-y\geqslant 0$.

$$\lim_{L\to\infty}\mathbb{E}_\Lambda^{per}[\phi_x\phi_y] = \tfrac{1}{4\pi}\int_{-\pi}^{\pi}\frac{e^{ik(x-y)}}{2\beta(1-\cos k)+m^2}dk = \tfrac{-i}{4\pi}\int_C \frac{z^{x-y}}{\beta(2-z-z^{-1})+m^2}\frac{dz}{z}$$
$$= \tfrac{-i}{4\pi}\int_C\frac{z^{x-y}}{\beta(2z-z^2-1)+m^2z}dz = \tfrac{-i}{4\pi\beta}\int_C\frac{z^{x-y}}{(z-z_1)(z_2-z)}dz$$

where $C = \{z = e^{i\theta} \in \mathbb{C} | \theta \in [0, 2\pi[\}$ is the circle of radius 1 and

$$z_1 = \left(1 + \tfrac{m^2}{2\beta}\right) - \sqrt{\left(1 + \tfrac{m^2}{2\beta}\right)^2 - 1} = \left(1 + \tfrac{m^2}{2\beta}\right) - \tfrac{\alpha}{\beta} < 1,$$
$$z_2 = \left(1 + \tfrac{m^2}{2\beta}\right) + \sqrt{\left(1 + \tfrac{m^2}{2\beta}\right)^2 - 1} = \left(1 + \tfrac{m^2}{2\beta}\right) + \tfrac{\alpha}{\beta} > 1$$

and $\alpha = \sqrt{(\beta + m^2/2)^2 - \beta^2}$ was introduced in Chapter 2. Since $x - y \geqslant 0$ the function inside the integral is holomorphic on the whole plane $\mathbb{C}$ except at the two points $z = z_1, z_2$ where it has a simple pole. Therefore

$$\lim_{L \to \infty} \mathbb{E}_{\Lambda}^{per}[\phi_x \phi_y] = \frac{-i2\pi i}{4\pi\beta} \frac{z^{x-y}}{z_2 - z_1} = \frac{1}{4\alpha} z_1^{|x-y|}$$

■

3.2 Gaussian integrals is $d \geqslant 1$.

3.2.1 The harmonic cristal in $d \geqslant 1$.

For $d \geqslant 1$ we consider the cube $\Lambda_L = \{-L, \ldots, L\}^d$. The set of possible configurations is now $\Omega_\Lambda = \{\phi : \Lambda \to \mathbb{R}\}$. The energy associated to a configuration is

$$\beta H_{\Lambda}^{(bc)}(\phi) = \sum_{j \sim k \in \Lambda} \beta(\phi_j - \phi_k)^2 + \sum_{j \in \Lambda} m^2 \phi_j^2 + F^{(b.c)}(\phi)$$

where $j \sim k$ is $\|j - k\| = 1$ (with the euclidian norm $\|x\|^2 = \sum_{\rho=1}^{d} x_\rho^2$) and

$$F^{(b.c)}(\phi) = \begin{cases} \sum_{\substack{z,z' \in \partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta(\phi_z - \phi_{z'})^2 & \text{periodic b.c.} \\ \sum_{\substack{z \in \partial\Lambda, z' \in \Lambda^c \\ \|z-z'\|=1}} \beta(\phi_z - \phi_{z'})^2_{\phi_{z'}=0} = \sum_{\substack{z \in \partial\Lambda, z' \in \Lambda^c \\ \|z-z'\|=1}} \beta\phi_z^2 & \text{Dirichlet b.c.} \\ 0 & \text{Neuman b.c.} \end{cases}$$

where $\|z - z'\|_p$ is the norm on the periodic torus $\mathbb{Z}^d/\Lambda_L$. All these expressions can be written as quadratic forms

$$\beta H_{\Lambda}^{(bc)}(\phi) = (\phi, A_{\Lambda}^{(b.c.)}\phi), \qquad A_{\Lambda}^{(b.c.)} = -\beta\Delta_{\Lambda}^{(b.c.)} + m^2 Id_\Lambda.$$

where $-\Delta_\Lambda$ is the generalization of the discrete Laplacian to dimension $d \geqslant 1$. The formulas for Gaussian integrals generalize directly to any dimension. In particular

$$\mathbb{E}_{\Lambda}^{(bc)}[\phi_x \phi_y] = \tfrac{1}{2}(A_{\Lambda}^{(bc)})^{-1}_{xy}.$$

3.2.2 Periodic boundary conditions

In the case of periodic boundary conditions we can apply discrete Fourier transform (as in $d = 1$) to prove

$$\mathbb{E}^{(per)}_{\Lambda}[\phi_x\phi_y] = \tfrac{1}{2m^2|\Lambda|} + \tfrac{1}{2|\Lambda|}\sum_{\substack{n\in\Lambda\\ n\neq 0}} \frac{e^{i(k_n,(x-y))}}{2\beta\sum_{\rho=1}^d(1-\cos k_n^\rho)+m^2}$$

where $n = (n_1,\dots n_d) \in \Lambda$, $k_n = (k_n^1,\dots,k_n^d)$ and $k_n^\rho = \frac{2\pi n_\rho}{2L+1}$. By the same arguments we used in $d = 1$ we can show that for small m

$$\left|(A^{(per)}_{\Lambda})^{-1}_{xy} - \tfrac{1}{m^2|\Lambda|}\right| = \begin{cases} O\left(\frac{1}{m}\right) & d=1\\ O\left(|\ln m|\right) & d=2 \\ O(1) & d\geqslant 3\end{cases} \tag{3.2.5}$$

the main reason being that for small n ie $\|n\| \leqslant L/10$ the Fourier sum can be approximated by the integral

$$\tfrac{1}{|\Lambda|}\sum_{\|n\|\leqslant L/10}\frac{1}{2\beta\sum_{\rho=1}^d(1-\cos k_n^\rho)+m^2} \sim \int_{\|k\|\leqslant \pi/10}\frac{1}{\|k\|^2+m^2}d^dk = C_d\int_0^{\pi/10}\frac{k^{d-1}}{k^2+m^2}dk.$$

This integral is linearly divergent in $d = 1$, log divergent in $d = 2$ and bounded in $d \geqslant 3$.

Infinite volume

As in $d = 1$ when $L \to \infty$ and m is kept fixed the Riemann sum converges to an integral

$$\lim_{\Lambda\to\mathbb{Z}^d}\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] = \frac{1}{2(2\pi)^d}\int_{[-\pi,\pi]^d}\frac{e^{i(k,(x-y))}}{2\beta\sum_{\rho=1}^d(1-\cos k^\rho)+m^2}d^dk.$$

With some extra work one can show that the limit exists also if we let $m \to 0$ and $L \to \infty$ simultaneously as long as $mL \to \infty$. Precisely we have

$$\lim_{\substack{m\to 0, L\to\infty\\ mL\to\infty}} c(m)\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] = \lim_{m\to 0}\lim_{L\to\infty} c(m)\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y]$$

where

$$c(m) = \begin{cases} m & d=1\\ |\ln m| & d=2\\ 1 & d\geqslant 3\end{cases}$$

To prove this result one has to compare the Riemann sum with the integral. The difference can be expressed as sum over gradients $f(k) - f(k_n)$ which in turn give some decay improvement by the same arguments we used to prove eq. (3.1.4).

3.2.3 Existence and uniqueness of the thermodynamic limit.

Theorem 4 *The thermodynamic limit for the 2 point correlation function* $\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y]$ *exists* $\forall d \geqslant 1$ *and is independent of the boundary conditions:*

$$\lim_{L\to\infty}\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{D}_{\Lambda,m}[\phi_x\phi_y]\right] = \lim_{L\to\infty}\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{N}_{\Lambda,m}[\phi_x\phi_y]\right] = 0,$$

for any fixed $m > 0$. *This remains true also is we let* $m \to 0$ *and* $L \to \infty$ *simultaneously with* $mL \to \infty$. *Precisely*

$$\lim_{\substack{m\to 0,\, L\to\infty \\ mL\to\infty}} c(m)\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{D}_{\Lambda,m}[\phi_x\phi_y]\right] = \lim_{\substack{m\to 0,\, L\to\infty \\ mL\to\infty}} c(m)\left[\mathbb{E}^{(per)}_{\Lambda,m}[\phi_x\phi_y] - \mathbb{E}^{N}_{\Lambda,m}[\phi_x\phi_y]\right] = 0.$$

Proof. Existence follows directly from the results of the previous section in the case of periodic boundary conditions. To prove uniqueness let $M^D = -\beta\Delta^D_\Lambda + m^2$ and $M^N = -\beta\Delta^N_\Lambda + m^2$ and $M = -\beta\Delta^{per}_\Lambda + m^2$ the matrices corresponding to Dirichlet, Neuman and periodic boundary conditions. We remark that M^D and M differ only on the boundary of Λ. The same is true for M^N. Precisely

$$M^D = M + X, \qquad M^N = M + \tilde{X}$$

where

$$X_{xy} = \sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta[\delta_{x,z}\delta_{y,z'} + \delta_{x,z'}\delta_{y,z}],$$

$$\tilde{X}_{xy} = \sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta[\delta_{x,z}\delta_{y,z'} + \delta_{x,z'}\delta_{y,z} - \delta_{x,z}\delta_{y,z} - \delta_{x,z'}\delta_{y,z'}].$$

For any two matrices A and B (with A and $A+B$ invertible) we have

$$(A+B)^{-1} - A^{-1} = -(A+B)^{-1}BA^{-1}.$$

Applying the relation above

$$\begin{aligned}
(M^D)^{-1}_{xy} - M^{-1}_{xy} &= (M+X)^{-1}_{xy} - M^{-1}_{xy} = -\sum_{zz'}(M+X)^{-1}_{xz}X_{zz'}M^{-1}_{z'y} \\
&= -\sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta\left[(M^D)^{-1}_{xz}M^{-1}_{z'y} + (M^D)^{-1}_{xz'}M^{-1}_{zy}\right] \\
(M^N)^{-1}_{xy} - M^{-1}_{xy} &= (M+\tilde{X})^{-1}_{xy} - M^{-1}_{xy} = -\sum_{zz'}(M+\tilde{X})^{-1}_{xz}\tilde{X}_{zz'}M^{-1}_{z'y} \\
&= -\sum_{\substack{z,z'\in\partial\Lambda \\ \|z-z'\|>1 \|z-z'\|_p=1}} \beta\left[(M^N)^{-1}_{xz} - (M^N)^{-1}_{xz'}\right]\left[M^{-1}_{zy} - M^{-1}_{z'y}\right]
\end{aligned}$$

Case of fixed mass. By Combes-Thomas estimate (see the next subsection) there exists a constant $\mu_{m,d}$ that depends only on m and d such that, for any boundary condition

$$\left|[M^{bc}]^{-1}_{xy}\right| \leqslant \tfrac{2}{m^2} e^{-|x-y|\mu_{m,d}}$$

uniformly in the volume. Then

$$\left|(M^D)^{-1}_{xy} - M^{-1}_{xy}\right| \leqslant \tfrac{CL^{d-1}}{m^4} e^{-\mu_{m,d} L} \rightarrow_{L\rightarrow\infty} 0$$

for some constant C. The same holds for M^N.

Case of vanishing mass with Dirichlet b.c. For m small the factor $\mu_m \simeq m^2$ so the Combes-Thomas estimate gives a decay in Lm^2 which is not enough to prove the convergence. By matrix-tree thorem (see the next subsection) we can prove

$$(M^D)^{-1}_{xy} \geqslant 0 \quad \forall x, y \in \Lambda, \text{ and } \quad \sum_{z\in\partial\Lambda} (M^D)^{-1}_{xz} \leqslant \tfrac{1}{\beta} \ \forall x \in \Lambda.$$

Moreover, by Fourier analysis (see eq. (3.2.5) above) one can show that

$$c(m)\left|M^{-1}_{xy}\right| \rightarrow_{\substack{m\rightarrow 0, L\rightarrow\infty \\ mL\rightarrow\infty}} 0$$

Putting together these estimates

$$c(m)\left|(M^D)^{-1}_{xy} - M^{-1}_{xy}\right| \leqslant \beta \sum_{z\in\partial\Lambda} (M^D)^{-1}_{xz}\ c(m) \sup_{z\in\partial\Lambda} \left|M^{-1}_{zy}\right| \rightarrow_{\substack{m\rightarrow 0, L\rightarrow\infty \\ mL\rightarrow\infty}} 0.$$

■

3.2.4 Combes-Thomas estimate

Theorem 5 (Combes-Thomas) *Let Γ be a finite or countable set, $M = T + U$ a self-adjoint operator on $l^2(\Gamma)$, with U an arbitrary diagonal operator and T an off-diagonal operator. Let $|x-y|$ the distance in Γ. If there exists a parameter $\eta > 0$ such that*

$$\sup_{x\in\Gamma} \sum_{y\in\Gamma} |T_{xy}| e^{\eta|x-y|} = S < \infty$$

then for any E outside the spectrum of M with dist$\{M, E\} = \Delta > 0$

$$\left|(M-E)^{-1}_{xy}\right| \leqslant \frac{2}{\Delta} e^{-\mu|x-y|}, \qquad \text{with } \mu = \frac{\Delta\eta}{\Delta + 2S}.$$

Proof. Let $e_x \in l^2(\Gamma)$ the function defined by $e_x(y) = \delta_{x=y}$, then

$$(M-E)^{-1}_{xy} = (e_x, (M-E)^{-1} e_y).$$

Let $R : l^2(\Gamma) \to l^2(\Gamma)$ the multiplication operator defined by

$$[Rf](y) = e^{\mu|x-y|_N} f(y), \qquad \text{where } |x-y|_N = \min\{|x-y|, N\}.$$

The parameter N makes R a bounded operator also when Γ is a countable set. At the end of the proof, we will take N to infinity. Then

$$(M-E)^{-1}_{xy}\ e^{\mu|x-y|_N} = (e_x, (M-E)^{-1} e_y)\ e^{\mu|x-y|_N} = (R^{-1}e_x,\ (M-E)^{-1}\ R\ e_y)$$
$$= (e_x,\ R^{-1}(M-E)^{-1}R\ e_y) = (e_x, \left[R^{-1}(M-E)R\right]^{-1} e_y)$$

Then

$$\left|(M-E)^{-1}_{xy}\right|\ e^{\mu|x-y|_N} \leqslant \left\| \frac{1}{R^{-1}(M-E)R} \right\| = \left\| \frac{1}{[R^{-1}TR-T]+[M-E]} \right\|$$

where we used $R^{-1}UR = U$. The kernel of $[R^{-1}TR-T]$ is given by $k(x,y)$

$$[R^{-1}TR-T]f(y) = \sum_z T_{yz}\left[e^{\mu(|z-x|_N - |y-x|_N)} - 1\right] f(z)$$
$$= \sum_z k(y,z) f(z).$$

Since $\big||z-x|_N - |y-x|_N\big| \leqslant |y-z|_N$ we have

$$\left|e^{\mu(|z-x|_N-|y-x|_N)} - 1\right| \leqslant \max\left[(e^{\mu|z-y|_N}-1), (1-e^{-\mu|z-y|_N})\right]$$
$$= e^{\mu|z-y|_N} - 1.$$

Then the kernel $k(y,z)$ satisfies

$$\sup_y \sum_z |k(y,z)| = \sup_z \sum_y |k(y,z)| \leqslant\ \sup_y \sum_z |T_{yz}| \left(e^{\mu|z-y|_N} - 1\right)$$
$$\leqslant \left[\sup_u e^{-\eta|u|}\left(e^{\mu|u|}-1\right)\right] \sup_y \sum_z |T_{yz}| e^{\eta|z-y|_N}$$
$$\leqslant S\tfrac{\mu}{\eta-\mu}\left(\tfrac{\eta-\mu}{\eta}\right)^{\frac{\eta}{\mu}} \leqslant S\tfrac{\mu}{\eta-\mu}.$$

since $\mu < \eta$. Then by the Schur's bound we have

$$\|[R^{-1}TR-T]\| \leqslant S\tfrac{\mu}{\eta-\mu} = \tfrac{\Delta}{2} \quad \text{since } \mu = \tfrac{\Delta\eta}{\Delta+2S}.$$

On the other hand

$$\|[M-E]f\| \geqslant \Delta\|f\| \quad \forall f \in l_2(\Gamma).$$

With these bounds we obtain

$$\left\|\tfrac{1}{[R^{-1}TR-T]+[M-E]}\right\| = \frac{1}{\inf_f \frac{\|[R^{-1}TR-T]f+[M-E]f\|}{\|f\|}} \leqslant \tfrac{2}{\Delta}$$

since

$$\|[R^{-1}TR-T]f + [M-E]f\| \geqslant \big|\|[R^{-1}TR-T]f\| - \|[M-E]f\|\big|.$$

These bounds do not depend on the N, so we can take $N \to \infty$. This completes the proof. ■

Application of Combes-Thomas: bound on the two point function. Let $A_\Lambda = -\beta\Delta_\Lambda^{(b.c)} + m^2 I_\Lambda$. For any choice of the boundary conditions we can write

$$A_\Lambda = T + U, \quad \text{where } |T_{x,y}| = \beta\delta_{|x-y|=1},$$

where $|x-y|$ is the euclidean norm in $\mathbb{Z}^d$. In the case of periodic boundary conditions $|x-y|$ is the euclidean norm on the torus $\mathbb{Z}^d/\Lambda_L$. Moreover $\|A\| \geqslant m^2$ and

$$\sum_y \sum_z |T_{z,y}| e^{\eta|z-y|} \leqslant 2d\beta e^\eta = S < \infty$$

for any choice of $\eta > 0$ and for any choice of the boundary conditions. Then we can apply Combes-Thomas estimate with $E = 0$

$$\left|\mathbb{E}_\Lambda^{(b.c.)}[\phi_x\phi_y]\right| = \frac{1}{2}|(A_\Lambda^{-1})_{xy}| \;\leqslant\; \tfrac{1}{m^2} e^{-\mu_m|x-y|}$$

where

$$\mu_m = \frac{m^2\eta}{m^2 + 4d\beta e^\eta} \tag{3.2.6}$$

and $\eta > 0$ is arbitrary. This bound holds uniformly in the volume Λ and for **any dimension** $d \geqslant 1$.

3.2.5 Matrix-tree theorem

Let Λ be a finite set of points. Let $E_\Lambda = \{(i,j)|\ i,j \in \Lambda, i \neq j\}$ be the set un **undirected edges** $e = (i,j) = (j,i)$ on Λ. For each edge $e \in E_\Lambda$ we denote its endpoints by i_e, j_e.

Definition 2 *A subset $E \subset E_\Lambda$ of edges forms a loop (cycle) if we can order its edges $E = (e_1, \ldots, e_n)$ such that $i_{e_l} = j_{e_{l-1}}$, $\forall l = 2, \ldots n$ and $i_{e_1} = j_{e_n}$.*

Definition 3 *A **forest** F on Λ is a subset of E_Λ with no cycle. Let $\mathcal{F}[\Lambda]$ be the set of forests on Λ.*

Definition 4 *A **spanning tree** T on Λ is a forest on Λ such that for each pair $x, y \in \Lambda$ there exists a path in T connecting x to y. Precisely there exists a subset $\gamma_{xy}^T = (e_1, \ldots e_n) \subset T$ such that $i_{e_1} = x$, $j_{e_n} = y$ and $i_{e_l} = j_{e_{l-1}}$ $\forall l = 2, \ldots, n$.*

Characterization of a forest. A forest F can be uniquely determined by the following information.

1. We fix a partition P of the set Λ.
2. Inside each element X of the partition we choose a spanning tree.

The forest is then obtained taking the union over the spanning trees. Note that this implies there is no edge connecting points in different elements of the partition. On the contrary any two points inside $X \in P$ are connected by a path in the forest. The elements $X \in P$ are also called *connected components* of the forest. For each forest F we denote by $P(F)$ the corresponding partition.

Theorem 6 (matrix-tree) *Let M be a $N \times N$ symmetric invertible matrix (not necessarily positive or real). Let $\Lambda = \{1, \ldots, N\}$. Then*

$$\det M = \sum_{F \in \mathcal{F}[\Lambda]} \prod_{e \in F} [-M_{i_e j_e}] \prod_{X \in P(F)} [\sum_{r \in X} B_r]$$

$$M_{xy}^{-1} \det M = \sum_{F \in \mathcal{F}_{xy}[\Lambda]} \prod_{e \in F} [-M_{i_e j_e}] \prod_{X \in P(F), x \notin X} [\sum_{r \in X} B_r]$$

where

$$B_r = \sum_{j \in \Lambda} M_{rj}$$

and $\mathcal{F}_{xy}[\Lambda]$ is the set of forests such that x and y belong to the same connected component. Alternatively one may write

$$\det M = \sum_{P \in \mathcal{P}[\Lambda]} \prod_{X \in P} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-M_{i_e j_e}] [\sum_{r \in X} B_r] \right\}$$

$$M_{xy}^{-1} \det M = \sum_{P \in \mathcal{P}_{xy}[\Lambda]} \left[\sum_{T \in \mathcal{T}[X_x]} \prod_{e \in T} [-M_{i_e j_e}] \right] \prod_{\substack{X \in P \\ x \notin X}} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-M_{i_e j_e}] [\sum_{r \in X} B_r] \right\}$$

where $\mathcal{P}[\Lambda]$ is the set partitions of Λ, $\mathcal{T}[X]$ the set of spanning trees on X, and finally $\mathcal{P}_{xy}[\Lambda]$ is the set of partitions such that x and y belong to the same element of the partition: this special element of the partition is denoted by X_x.

Remark. The general matrix-tree theorem applies also to non-symmetric and non invertible matrices, with a slight modification in the definitions.

With these definitions we can prove the following result.

Lemma 13 *Let $\Lambda = \{-L, \ldots, L\}^d$ and $A_\Lambda = -\beta \Delta_\Lambda^D + m^2 \mathbb{I}_\Lambda$ a matrix on $\Lambda \times \Lambda$, where $-\Delta_\Lambda^D$ is the discrete Laplacian with Dirichlet boundary conditions. Then*

$$0 \leqslant (A_\Lambda^{-1})_{xy} \qquad \forall x, y \in \Lambda \tag{3.2.7}$$

and

$$\sum_{z \in \partial\Lambda} (A_\Lambda^{-1})_{xz} \leqslant \frac{1}{\beta} \qquad \forall x \in \Lambda. \tag{3.2.8}$$

Proof. Applying the matrix-tree theorem we can write

$$(A_\Lambda^{-1})_{xy} = \frac{\sum_{P \in \mathcal{P}_{xy}[\Lambda]} \left[\sum_{T \in \mathcal{T}[X_x]} \prod_{e \in T} [-A_{i_e j_e}] \right] \prod_{\substack{X \in P \\ x \notin X}} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-A_{i_e j_e}] [\sum_{r \in X} B_r] \right\}}{\sum_{P \in \mathcal{P}[\Lambda]} \prod_{X \in P} \left\{ \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [-A_{i_e j_e}] [\sum_{r \in X} B_r] \right\}}$$

Note that $-A_{i_e j_e} = \beta$ when $i_e \sim j_e$, i.e $|i_e - j_e| = 1$ and zero otherwise (since $i_e \neq j_e$ for any edge e in the forest). Then only nearest neighbor edges give a non zero contribution. Let $\mathcal{P}^c[\Lambda]$ the set of partitions of Λ into *connected components* and $\tilde{\mathcal{T}}[X]$ the set of trees on X made only of nearest neighbor pairs $i \sim j$. Finally note that

$$B_r = \sum_{j \in \Lambda} A_{jr} = m^2 + \beta d_r$$

where

$$d_r = \#\{j \in \Lambda^c | \; |j - r| = 1\} \quad \text{so} \quad \begin{cases} d_r = 0 & \text{if } r \in \Lambda \backslash \partial\Lambda \\ d_r \in \{1, \ldots, d\} & \text{if } r \in \partial\Lambda. \end{cases}$$

Then $\sum_{r \in X} B_r = m^2|X| + \beta d_X$, where

$$d_X = \sum_{r \in X} d_r, \qquad \text{hence} \quad |X \cap \partial\Lambda| \leqslant d_X \leqslant d|X \cap \partial\Lambda|.$$

Inserting all this we obtain

$$(A_\Lambda^{-1})_{xy} = \frac{\sum_{P \in \mathcal{P}^c_{xy}[\Lambda]} \left[\sum_{T \in \tilde{\mathcal{T}}[X_x]} \beta^{|T|}\right] \prod_{\substack{X \in P \\ x \notin X}} \left\{\sum_{T \in \tilde{\mathcal{T}}[X]} \beta^{|T|} \; [m^2|X| + \beta d_X]\right\}}{\sum_{P \in \mathcal{P}^c[\Lambda]} \prod_{X \in P} \left\{\sum_{T \in \tilde{\mathcal{T}}[X]} \beta^{|T|} \; [m^2|X| + \beta d_X]\right\}}$$

This expression is manifestly positive hence (3.2.7). Let

$$\omega(X) = [m^2|X| + \beta d_X] \sum_{T \in \tilde{\mathcal{T}}[X]} \beta^{|T|}.$$

Then

$$\rho(P) = \frac{\prod_{X \in P} \omega(X)}{\sum_{P \in \mathcal{P}^c[\Lambda]} \prod_{X \in P} \omega(X)}$$

is a probability measure on $\mathcal{P}^c[\Lambda]$ and $(A_\Lambda^{-1})_{xy}$ can be expressed as an average

$$(A_\Lambda^{-1})_{xy} = \sum_{P \in \mathcal{P}^c_{xy}[\Lambda]} \rho(P) \tfrac{1}{[m^2|X_x| + \beta d_{X_x}]}$$

To prove (3.2.8) we replace y by z and sum over all $z \in \partial\Lambda$

$$\begin{aligned} \sum_{z \in \partial\Lambda} (A_\Lambda^{-1})_{xz} &= \sum_{z \in \partial\Lambda} \sum_{P \in \mathcal{P}^c_{xz}[\Lambda]} \rho(P) \tfrac{1}{[m^2|X_x| + \beta d_X]} \\ &= \sum_{P \in \mathcal{P}^c_{x\partial\Lambda}[\Lambda]} \sum_{z \in X_x \cap \partial\Lambda} \rho(P) \tfrac{1}{[m^2|X_x| + \beta d_{X_x}]} = \sum_{P \in \mathcal{P}^c_{x\partial\Lambda}[\Lambda]} \rho(P) \tfrac{|X_x \cap \partial\Lambda|}{[m^2|X_x| + \beta d_{X_x}]} \leqslant \frac{1}{\beta} \end{aligned}$$

since $d_{X_x} \geqslant |X_x \cap \partial\Lambda|$. This ends the proof. ■

3.3 Perturbation around a gaussian integral

3.3.1 The $O(n)$ model

Let $\Lambda = \{-L, \dots, L\}^d$ a cube inside $\mathbb{Z}^d$. To each lattice point j we associate a spin $S_j \in \mathcal{S}_n$ taking values in the unit $n-$dimensional sphere. The three main examples are

1. $n = 1$: in this case $S_j = \pm 1$ and we obtain the Ising model;
2. $n = 2$: the spin takes values on the unit circle. This is the so called XY (or rotator) model;
3. $n = 3$: the spin takes value on the sphere. This is the so called Heisenberg model.

The space of configurations is $\Omega_\Lambda = \{S : \Lambda \to \mathcal{S}_n\}$ and the corresponding Gibbs measure is

$$d\mu_{\Lambda,n}^{\beta,h}(S) = \prod_{j\in\Lambda} d\Omega_n(S_j)\; e^{\frac{\beta}{2}\sum_{j,k\Lambda} J_{jk}(S_j,S_k)} e^{(h,\sum_{j\in\Lambda} S_j)} \tag{3.3.9}$$

where $(.,.)$ is the euclidean scalar product in $\mathbb{R}^n$, $h \in \mathbb{R}^n$ is the magnetic field and J_{jk} is a collection of real interaction constants such that

$$J_{jk} = J_{kj} \geqslant 0 \;\forall\; j,k \in \Lambda$$

and there exists a constant $c > 0$ independent of the volume Λ such that

$$0 \leqslant \sum_{k\in\Lambda} J_{jk} \leqslant c \qquad \forall j \in \Lambda.$$

One can understand this constraint by regarding J_{jk} as the probability to jump from j to k. Then $\sum_{k\in\Lambda} J_{jk} = 1$ since it is the probability of jumping to *any* point. Finally $d\Omega_n$ is the invariant measure on the sphere $\mathcal{S}_n$, normalized to 1. In particular

1. for the Ising model the measure is discrete: $\int d\Omega_1 = \frac{1}{2}\sum_{\sigma=\pm1}$;
2. for $n = 2$ we can parametrize the circle by one angle: $\int d\Omega_2 = \frac{1}{2\pi}\int_0^{2\pi} d\theta$;
3. for $n = 3$ we can parametrize the sphere by two angles: $\int d\Omega_3 = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta$.

Phenomenology and symmetries.

Since $J_{jk} \geqslant 0$ the interaction favors the configurations with spins *aligned* (we have a so called "*ferromagnetic interaction*").

When $h = 0$ the Gibbs measure is invariant under global rotation

$$S_j \to US_j \;\forall j \qquad U^*U = \mathrm{Id}_{\mathbb{R}^n} \tag{3.3.10}$$

for any $n \geqslant 2$. In particular it is invariant under *flip* $S_j \to -S_j$ $\forall j$ (this is true also for $n = 1$). Then

$$\mathbb{E}_\Lambda^{\beta,h=0}[S_j] = -\mathbb{E}_\Lambda^{\beta,0}[S_j] \Rightarrow \ \mathbb{E}_\Lambda^{\beta,0}[S_j] = 0 \ \forall j \in \Lambda, \ \forall n \geqslant 1, \ \forall d \geqslant 1,$$

and we say that the average magnetization is zero (the spins are not aligned). For general h, the finite volume magnetization $\mathbb{E}_\Lambda^{\beta,h}[\frac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j]$ is a smooth function in each component of the vector h hence

$$\lim_{\Lambda\to\mathbb{Z}^d} \ \lim_{h\to 0} \mathbb{E}_\Lambda^{\beta,h} \left[\frac{1}{|\Lambda|} \sum_{j\in\Lambda} S_j \right] = 0.$$

If we invert the limits we may have two results:

$$\lim_{h\to 0} \mathbb{E}_\Lambda^{\beta,h} \ \lim_{\Lambda\to\mathbb{Z}^d} \mathbb{E}_\Lambda^{\beta,h} \left[\frac{1}{|\Lambda|} \sum_{j\in\Lambda} S_j \right] = \begin{cases} 0 \\ M \neq 0. \end{cases}$$

In the first case there is no magnetization. This means the infinite volume measure $\lim_{\Lambda\to\mathbb{Z}^d} d\mu_{\Lambda,\beta,h}$ recovers the flip symmetry when $h \to 0$. In this case we say *the symmetry is restored*. In the second case we have magnetization. Then the infinite volume measure $\lim_{\Lambda\to\mathbb{Z}^d} d\mu_{\Lambda,\beta,h}$ does not recover the symmetry when $h \to 0$. Then we say we have *spontaneous symmetry breaking*.

One can show that at high enough temperature (i.e. β small) there is never a magnetization, since the thermal fluctuations are too strong. On the contrary at low temperature (i.e. β large) the forces trying to align the spins may be strong enough to create a magnetization. In this case we say we have a *phase transition*.

Mermin-Wagner: low dimensional systems.

Phase transitions are harder to observe in low dimensions. This is the content of the so called *Mermin-Wagner theorem* (also known a Mermin-Wagner-Hohenberg theorem or Coleman theorem). It is a series of papers that can be summarized in the following statement:

Continuous symmetries cannot be spontaneously broken at finite temperature in systems with sufficiently short-range interactions in dimensions $d \leqslant 2$.

Application to $O(n)$ with short range interaction.

Let us consider the $O(n)$ model defined above with $J_{jk} = 1$ when $|j - k| = 1$ and $J_{jk} = 0$ otherwise. Then

$$d\mu(S) = \prod_{j\in\Lambda} d\Omega_n(S_j) \ e^{\beta \sum_{j\sim k}(S_j,S_k)} e^{(h,\sum_{j\in\Lambda} S_j)},$$

where $j \sim k$ means the two points are nearest neighbors in $\mathbb{Z}^d$. For $n \geqslant 2$ this measure has a continuous symmetry at $h = 0$, so by Mermin-Wagner theorem we cannot expect a magnetization (hence a phase transition) in $d \leqslant 2$. The theorem does not apply to $n = 1$ (Ising model) since there the symmetry is discrete ($\sigma \to -\sigma$).

In $d = 2$ one may still observe a softer version of phase transition known as *Kosterlitz-Thouless transition* that corresponds to a change in the decay rate of two point correlations. More precisely

$$\lim_{\Lambda \to \mathbb{Z}^d} \mathbb{E}_\Lambda^{\beta,0} \ [S_0 S_x] \ = \begin{cases} c_1 e^{-\frac{|x|}{\xi}} & T \gg 1 (\text{i.e.} \beta \ll 1) \\ \frac{c_2}{|x|^\eta} & T \ll 1 (\text{i.e.} \beta \gg 1) \end{cases} \qquad \text{as } |x| \gg 1.$$

for some constants $c_1, c_2, \xi, \eta > 0$.

3.3.2 A first example of perturbation around a Gaussian measure: the $O(2)$ model in $d = 2$

Let $\Lambda = \mathbb{Z}^2 / \{-L, \ldots, L\}^2$ a cube in $\mathbb{Z}^2$ with periodic boundary conditions. The space of configurations is $\Omega_\Lambda = \{S : \Lambda \to \mathcal{S}_2\}$ and we consider the Gibbs measure

$$d\mu(S) = \prod_{j \in \Lambda} d\Omega_2(S_j) \ e^{\beta \sum_{j \sim k} (S_j, S_k)}$$

where $j \sim k$ are pairs at distance one in the torus. For this model one can prove a Kosterlitz-Thouless transition. More precisely we have

Theorem 1 [Mc Bryan, Spencer (1977)]. For any $0 < \epsilon < 1$ there exists a $\beta_0(\epsilon) > 0$ such that for all $\beta \geqslant \beta_0(\epsilon)$

$$\lim_{\Lambda \to \mathbb{Z}^d} |\mathbb{E}_\Lambda^{\beta,0} [S_0 S_x]| \ \leqslant \frac{1}{|x|^{\frac{1-\epsilon}{2\pi\beta}}} \tag{3.3.11}$$

Theorem 2 [Fröhlich, Spencer (1981)]. There exists a a $\beta_0 > 0$ and a constant $c > 0$ such that for all $\beta \geqslant \beta_0$

$$\lim_{\Lambda \to \mathbb{Z}^d} |\mathbb{E}_\Lambda^{\beta,0} [S_0 S_x]| \ \geqslant \frac{c}{|x|^{\frac{1}{2\pi\beta}}}.$$

Theorem 3. There exists a a $\beta_0 > 0$ such that for all $\beta \leqslant \beta_0$

$$\lim_{\Lambda \to \mathbb{Z}^d} |\mathbb{E}_\Lambda^{\beta,0} [S_0 S_x]| \ \leqslant C_\beta e^{-\frac{|x|}{\xi_\beta}}$$

In this section we will review the proof of Theorem 1. This is based on two steps. The first is non rigorous and consists in approximating the measure by a Gaussian integral. The second step is rigorous and consists in mimicking some of the operations we did to compute the (non-rigorous) Gaussian approximation in a rigorous context. The key step is a complex deformation.

Proof of Theorem 1 (based on [Mc Bryan, Spencer]) Using polar coordinates the average above can be written as

$$\begin{aligned}\mathbb{E}_{\Lambda}^{\beta,0}\left[S_0 S_x\right] &= \frac{1}{Z_\Lambda}\int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})}\cos(\theta_x-\theta_0)\prod_{j\in\Lambda} d\theta_j \\ &= \frac{1}{2Z_\Lambda}(I_+ + I_-)\end{aligned}$$

where the partition function is

$$Z_\Lambda = \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})}\prod_{j\in\Lambda} d\theta_j$$

and we defined

$$I_\sigma = \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})} e^{i\sigma(\theta_x-\theta_0)}\prod_{j\in\Lambda} d\theta_j, \quad \sigma = \pm 1.$$

Preliminary heuristic arguments. Using some *non rigorous arguments* we establish what kind of behavior we expect from the integrals above. Since $1 \geqslant \cos(\theta_j - \theta_{j'}) \geqslant -1$ and $\beta \gg 1$, the function $\exp[\beta\cos(\theta_j - \theta_{j'})]$ is exponentially small unless $\theta_j - \theta_{j'} \simeq 0$ or 2π. Inspired by this fact we perform two approximations.

a). We take the Taylor expansion up to order 2 and neglect the remainder. Then

$$e^{\beta\sum_{j\sim j'}\cos(\theta_j-\theta_{j'})} \simeq e^{-\beta \mathrm{C}(\Lambda)} e^{-\frac{\beta}{2}\sum_{j\sim j'}(\theta_j-\theta_{j'})^2} = e^{-\beta \mathrm{C}(\Lambda)} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)}$$

where $C(\Lambda) = \sum_{j\sim j'} 1$ is a constant independent of θ and $-\Delta_\Lambda$ is the discrete Laplacian on Λ with periodic boundary conditions.

b). We replace the interval $[0, 2\pi]$ by $\mathbb{R}$ in the integral, for each $j \in \Lambda$. Inserting these two approximations both in the numerator and in the partition function above we obtain

$$\frac{I_\sigma}{Z_\Lambda} \simeq \frac{\int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)} e^{i\sigma(\theta_x-\theta_0)}\prod_{j\in\Lambda} d\theta_j}{\int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)}\prod_{j\in\Lambda} d\theta_j}$$

where the normalization is

$$\mathcal{N} = \int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{\beta}{2}(\theta,-\Delta_\Lambda\theta)}\prod_{j\in\Lambda} d\theta_j.$$

These two integrals are ill defined since $-\Delta_\Lambda$ is not invertible! One may give a sensible definition of a Gaussian measure even in this situation, but since here we are doing non rigorous arguments we ignore the problem. We introduce now the two functions

$$\begin{array}{llcll} v: & \Lambda \to \mathbb{R} & , & \alpha: & \Lambda \to \mathbb{R} \\ & j \to v_j = \delta_{jx} - \delta_{j0} & & & j \to \alpha_j = [(-\beta\Delta_\Lambda)^{-1} v]_j. \end{array} \tag{3.3.12}$$

Note that $\sum_j v_j = (1, v) = 0$, $v \in \ker(-\Delta_\Lambda)^\perp$, therefore the function α is *well defined*, even if $(-\Delta_\Lambda)$ is not invertible. Then $(\theta_x - \theta_0) = (v, \theta)$ and

$$I_\sigma = e^{-\frac{1}{2}(\alpha,(-\beta\Delta_\Lambda)\alpha)} \int_{\mathbb{R}^{|\Lambda|}} e^{-\frac{1}{2}((\theta - i\sigma\alpha), -\beta\Delta_\Lambda(\theta - i\sigma\alpha))} \prod_{j\in\Lambda} d\theta_j = e^{-\frac{1}{2}(\alpha,(-\beta\Delta_\Lambda)\alpha)}\mathcal{N}$$

where in the last step we perform the complex traslation

$$\theta_j \to \theta_j + i\sigma\alpha_j, \qquad \forall j \in \Lambda.$$

Now inserting the definition of α

$$\begin{aligned}
(\alpha, (-\beta\Delta_\Lambda)\alpha) &= (v, (-\beta\Delta_\Lambda)^{-1} v) \\
&= \frac{1}{\beta}\left[(-\Delta_\Lambda)^{-1}_{00} - (-\Delta_\Lambda)^{-1}_{0x} + (-\Delta_\Lambda)^{-1}_{x0} - (-\Delta_\Lambda)^{-1}_{xx}\right] \\
&= \frac{1}{\beta(2L+1)^2} \sum_{n\in\Lambda_L\setminus 0} \frac{2(1-\cos(k_n x))}{2(1-\cos(k_{n1})) + 2(1-\cos(k_{n2}))} \\
&= 2\frac{1}{2\pi\beta} \ln\|x\| \left[1 + O\left(\tfrac{1}{\ln\|x\|}\right)\right] \sim 2\frac{1}{2\pi\beta}\ln\|x\| \qquad |x| \gg 1
\end{aligned}$$

With these approximations we would obtain

$$\mathbb{E}^{\beta,0}_\Lambda[S_0 S_x] = \frac{I_+ + I_-}{2Z_\Lambda} \simeq \frac{1}{|x|^{\frac{1}{2\pi\beta}}}, \qquad |x| \gg 1.$$

Step 2. Inspired by the non rigorous arguments above we perform the following complex translation in the integral I_σ:

$$\theta_j \to \theta_j + i\sigma\alpha_j, \qquad \forall j \in \Lambda,$$

where α_j is defined in (3.3.12). Remember that the definitions given in (3.3.12) make sense even though $(-\Delta)$ is not invertible. The integral becomes

$$I_\sigma = e^{-(\alpha_x - \alpha_0)} \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j - \theta_{j'} + i(\alpha_j - \alpha_{j'}))} e^{i\sigma(\theta_x - \theta_0)} \prod_{j\in\Lambda} d\theta_j.$$

In order to close the contour in the complex plane we need to add the integrals along the paths $y_j = iz_j$, $z_j \in [0, \sigma\alpha_j]$ and $y_j = 2\pi + iz_j$, $z_j \in [0, \sigma\alpha_j]$. By periodicity they cancel each other. Since

$$\cos(\theta_j - \theta_{j'} + i(\alpha_j - \alpha_{j'})) = \cos(\theta_j - \theta_{j'})\cosh(\alpha_j - \alpha_{j'}) - i\sin(\theta_j - \theta_{j'})\sinh(\alpha_j - \alpha_{j'})$$

after inserting absolute values we have

$$\begin{aligned}
|I_\sigma| &\leqslant e^{-(\alpha_x - \alpha_0)} \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j - \theta_{j'})\cosh(\alpha_j - \alpha_{j'})} \prod_{j\in\Lambda} d\theta_j \\
&\leqslant e^{-(\alpha_x - \alpha_0)} e^{\beta\sum_{j\sim j'}[\cosh(\alpha_j - \alpha_{j'}) - 1]} \int_{[0,2\pi]^{|\Lambda|}} e^{\beta\sum_{j\sim j'}\cos(\theta_j - \theta_{j'})} \prod_{j\in\Lambda} d\theta_j \\
&= Z_\Lambda e^{-(\alpha_x - \alpha_0)} e^{\beta\sum_{j\sim j'}[\cosh(\alpha_j - \alpha_{j'}) - 1]}
\end{aligned}$$

where in the second line we use

$\cos(\theta_j - \theta_{j'})[\cosh(\alpha_j - \alpha_{j'}) - 1] + \cos(\theta_j - \theta_{j'}) \leqslant [\cosh(\alpha_j - \alpha_{j'}) - 1] + \cos(\theta_j - \theta_{j'})$.

Now

$$|\alpha_j - \alpha_{j'}| \leqslant \frac{1}{\beta} |[(-\Delta)^{-1}_{j0} - (-\Delta)^{-1}_{j'0}] + [(-\Delta)^{-1}_{jx} - (-\Delta)^{-1}_{j'x}]| \leqslant \frac{K}{\beta}$$

for some constant K independent of x and Λ. This last inequality can be obtained by the same kind of arguments in the Fourier sum we used to prove the estimate (3.1.2) in Lemma 11. Since β is large we can make $|\alpha_j - \alpha_{j'}|$ as small as we want. To complete the argument note that for any $0 < \epsilon < 1$ there exists a $\delta(\epsilon) > 0$ such that

$$\cosh(t) - 1 \leqslant \frac{1 + \epsilon/2}{2} t^2 \qquad \forall |t| \leqslant \delta,$$

where the factor 1/2 in front of ϵ is just a convenient choice to control some additional error terms later in the proof. From the bound above there exists a β_0 such that $|\alpha_j - \alpha_{j'}| \leqslant \delta$ for all $j \sim j'$ and for any $\beta \geqslant \beta_0$. Inserting this in our estimate we obtain

$$\begin{aligned} |\mathbb{E}^{\beta,0}_{\Lambda}\left[S_0 S_x\right]| &\leqslant \frac{|I_+| + |I_-|}{2 Z_\Lambda} \leqslant e^{-(\alpha_x - \alpha_0)} e^{\beta \sum_{j \sim j'} \frac{1+\epsilon/2}{2} (\alpha_j - \alpha_{j'})^2} \\ &= e^{-(\alpha_x - \alpha_0)} e^{\frac{1+\epsilon/2}{2} (\alpha, -\beta \Delta_\Lambda \alpha)} \\ &= e^{-\frac{1-\epsilon/2}{2} (v, (-\beta\Delta_\Lambda)^{-1} v)} = e^{-\frac{1-\epsilon/2}{2\pi\beta} \ln \|x\| \left[1 + O\left(\frac{1}{\ln \|x\|}\right)\right]} \\ &\leqslant e^{-\frac{1-\epsilon}{2\pi\beta} \ln \|x\|} = \frac{1}{|x|^{\frac{1-\epsilon}{2\pi\beta}}} \end{aligned}$$

where in the last line we use $(\alpha_x - \alpha_0) = (v, \alpha)$, $\alpha = (-\beta\Delta_\Lambda)^{-1} v$ and we take $\|x\|$ large enough to ensure

$$\left[1 + O\left(\tfrac{1}{\ln \|x\|}\right)\right] \geqslant (1 - \epsilon/2).$$

This concludes the proof.

3.3.3 An example of phase transition: the mean field case

In this section we consider the $O(n)$ model defined in (3.3.9) with non zero magnetic field $h \in \mathbb{R}^n$ and with interaction parameter

$$J_{jk} = \frac{1}{|\Lambda|} \qquad \forall i, j \in \Lambda.$$

With this choice

$$0 \leqslant \sum_{k \in \Lambda} J_{jk} \leqslant 1 \qquad \forall j \in \Lambda.$$

Note that in this case we have *long range interactions* since J_{jk} is constant for any pair $jk \in \Lambda$. Then the Mermin-Wagner theorem does not apply and one may have a phase transition also in $d = 2$.

Duality

The partition function in the mean field $O(n)$ model can be reformulated as an integral over n real variables

Lemma 14 *For any dimension $d \geqslant 1$ and any $n \geqslant 1$ we have*

$$Z^{\beta}_{\Lambda,n}(h) = \int d\mu^{\beta,h}_{\Lambda,n}(S) = \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-|\Lambda| F_{n,\beta}(x,h)}, \qquad \text{with } \mathcal{N}_{\Lambda,n,\beta} = \left(\frac{2\pi\beta}{|\Lambda|}\right)^{|\Lambda|},$$

$$\begin{array}{lll} F_{n,\beta}: & \mathbb{R}^n \times \mathbb{R}^n & \to \mathbb{R} \\ & (x,h) & \to F_{n,\beta}(x,h) = \frac{\|(x-h\|^2}{2\beta} - \ln J_n(\|(x\|) \end{array}$$

and

$$\begin{array}{llll} J_n: & \mathbb{R}^+ & \to \mathbb{R}^+ & \\ & t & \to J_n(t) = \cosh t & \text{if } n = 1 \\ & & \quad = \frac{\pi}{2}\int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2} \cosh[t\cos\theta] d\theta & \text{if } n \geqslant 2 \end{array}$$

Proof Since $J_{ij} = |\Lambda|^{-1}$ $\forall i, j$ we can write

$$e^{\frac{\beta}{2}\sum_{j,k\Lambda} J_{jk}(S_j,S_k)} = e^{\frac{\beta}{2|\Lambda|}\|\sum_{j\in\Lambda} S_j\|^2} = \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-\frac{|\Lambda|}{2\beta}\|x\|^2} e^{(x,\sum_{j\in\Lambda} S_j)}$$

Exchanging the integrals we obtain

$$\begin{aligned} Z^{\beta}_{\Lambda,n}(h) &= \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-\frac{|\Lambda|}{2\beta}\|x\|^2} \left[\int d\Omega_n(S)\; e^{(x+h,S)}\right]^{|\Lambda|} \\ &= \frac{1}{\mathcal{N}_{\Lambda,n,\beta}} \int_{\mathbb{R}^n} d^n x \; e^{-\frac{|\Lambda|}{2\beta}\|x-h\|^2} \left[\int d\Omega_n(S)\; e^{(x,S)}\right]^{|\Lambda|} \end{aligned}$$

When $n = 1$ we have

$$\int d\Omega_1(S)\; e^{(x,S)} = \frac{1}{2}\sum_{\sigma=\pm 1} e^{x\sigma} = \cosh(x) = \cosh(|x|).$$

When $n = 2$ we have

$$\int d\Omega_2(S)\, e^{(x,S)} = \frac{1}{2\pi}\int_0^{2\pi} e^{\|x\|\cos\theta} d\theta = \frac{1}{\pi}\int_0^{\pi} e^{\|x\|\cos\theta} d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \cosh(\|x\|\cos\theta) d\theta,$$

where in the first passage we perform a rotation in order to have x parallel to the vertical axis, then go to polar coordinates. Similarly for $n > 2$ we have

$$\int d\Omega_n(S)\, e^{(x,S)} = \frac{1}{\pi}\int_0^{\pi} (\sin\theta)^{n-2} e^{\|x\|\cos\theta} d\theta = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} (\sin\theta)^{n-2}\cosh(\|x\|\cos\theta) d\theta.$$

□

Remarks. The duality reduces the problem to the study of a n variable integral (compared to $n^{|\Lambda|}$ variables in the initial representation). Moreover, for large $|\Lambda|$ the integral will be concentrated around the minimal with respect to x of the function $F_n(x,h)$, therefore a saddle point analysis is possible.

Generating function

Using the dual representation above the average magnetization at finite volume can be expressed as

$$\mathbb{E}_{\Lambda}^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \tfrac{1}{|\Lambda|}\partial_h \ln Z_{\Lambda,n}^{\beta}(h) = \tfrac{1}{\beta}\frac{\int_{\mathbb{R}^n} d^n x\ (x-h)e^{-|\Lambda|F_{n,\beta}(x,h)}}{\int_{\mathbb{R}^n} d^n x\ e^{-|\Lambda|F_{n,\beta}(x,h)}} \qquad (3.3.13)$$

Phase transition

Theorem The $O(n)$ model in the mean field case has a phase transition in any $d \geqslant 1$. Precisely

$$\lim_{h\to 0+}\lim_{\Lambda\to\mathbb{Z}^d}\mathbb{E}_{\Lambda}^{\beta,h}\left[\frac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \begin{cases} 0 & \text{if } \beta<1 \text{ (high temperature)} \\ M_{d,\beta,n}>0 & \text{if } \beta>1 \text{ (low temperature)} \end{cases}$$

Proof In the following we set $h>0$. By (3.3.13) the problem can be reduced to the rigorous saddle analysis of a n variable integral. For simplicity we will restrict here to the case $n=1$. Then

$$F_1(x,h) = \frac{(x-h)^2}{2\beta} - \ln\cosh x$$

and the equations for the first and second derivative are

$$\partial_x F_1(x,h) = \tfrac{(x-h)}{\beta} - \tanh x, \qquad \partial_x^2 F_1(x,h) = \tfrac{1}{\beta} - \tfrac{1}{(\cosh x)^2}.$$

Note that

$$\partial_x^2 F_1(x,h) \leqslant \frac{1}{\beta} \qquad \forall x,h. \qquad (3.3.14)$$

Case 1: $\beta<1$ (high temperature). In this case F_1 is a convex function in x

$$\partial_x^2 F_1(x,h) \geqslant \tfrac{(1-\beta)}{\beta}\ \forall x,h \qquad (3.3.15)$$

therefore F_1 has only one minimum at the point $x_0(h)$ satisfying

$$\tfrac{(x_0-h)}{\beta} = \tanh x_0.$$

At $h=0$ $x_0=0$ is a solution of this equation, therefore $\lim_{h\to 0} x_0(\beta,h)=0$. By a Taylor expansion with integral remainder

$$F_1(x,h) = F_1(x_0,h) + (x-x_0)^2\int_0^1 (1-t)\partial_x^2 F_1(x_0+t(x-x_0),h)\ dt.$$

Inserting (3.3.14) and (3.3.15) we obtain $\forall x, h$

$$F_1(x_0,h) + \tfrac{1}{2\beta}(x-x_0)^2 \geqslant F_1(x,h) \geqslant F_1(x_0,h) + \tfrac{(1-\beta)}{2\beta}(x-x_0)^2. \qquad (3.3.16)$$

Now we can reexpress (3.3.13) as

$$\mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \tfrac{1}{\beta}\frac{\int_{\mathbb{R}} dx\ (x-h)e^{-|\Lambda|F_{1,\beta}(x,h)}}{\int_{\mathbb{R}} dx\ e^{-|\Lambda|F_{1,\beta}(x,h)}} = \tfrac{x_0(\beta,h)-h}{\beta} + R(\beta,h,|\Lambda|),$$

where

$$R(\beta,h,|\Lambda|) = \frac{\int_{\mathbb{R}} dx\ (x-x_0)e^{-|\Lambda|F_{n,\beta}(x,h)}}{\int_{\mathbb{R}} dx\ e^{-|\Lambda|F_{n,\beta}(x,h)}}.$$

Inserting absolute values, and the upper and lower estimates from (3.3.16) we obtain

$$\begin{aligned}|R(\beta,h,|\Lambda|)| &\leqslant \frac{\int_{\mathbb{R}} dx\ |x-x_0| e^{-\frac{|\Lambda|(1-\beta)}{2\beta}(x-x_0)^2}}{\int_{\mathbb{R}} dx\ e^{-\frac{|\Lambda|}{2\beta}(x-x_0)^2}} \\ &= \frac{2\int_0^\infty dx\ x\ e^{-\frac{|\Lambda|(1-\beta)}{2\beta}x^2}}{\int_{\mathbb{R}} dx\ e^{-\frac{|\Lambda|}{2\beta}x^2}} = \tfrac{1}{\sqrt{|\Lambda|}}\tfrac{2\sqrt{\beta}}{\sqrt{2\pi}(1-\beta)} \to_{|\Lambda|\to\infty} 0\end{aligned}$$

Finally

$$\lim_{h\to 0+}\lim_{\Lambda\to\mathbb{Z}^d}\mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \lim_{h\to 0+}\tfrac{x_0(\beta,h)-h}{\beta} = 0.$$

Case 2: $\beta > 1$ (low temperature). In this case the function $F_1(x,h)$ has two minimum points $x_1(h), x_2(h)$ satisfying

$$x_1(h) < 0, \quad x_2(h) > 0, \quad \lim_{h\to 0} x_2(h) = -\lim_{h\to 0} x_1(h) = x_0(\beta) > 0.$$

At $h=0$ F_1 is symmetric in x so the two minimums are at the same height

$$F_1(-x_0(\beta),0) = F_1(x_0(\beta),0) = F_m.$$

To see what is the approximate value of the two minimum points at $h \neq 0$, we expand near $h=0$ (remember that at the end we will take the limit $h \to 0$)

$$x_j(h) = \sigma_j x_0 + \delta_j h + O(h^2), \quad \sigma_1 = -1,\ \sigma_2 = 1.$$

Inserting this relation in the saddle point equation we obtain

$$\begin{aligned}0 &= \partial_x F_1(x_j(h),h) \\ &= \partial_x F_1(x_j(0),0)\ +\ \partial_x^2 F_1(x_j(0),0)\ \delta_j h\ +\ \partial_h\partial_x F_1(x_j(0),0)\ h\ +\ O(h^2) \\ &= h\left(\partial_x^2 F_1(x_j(0)\ \delta_j + \partial_h\partial_x F_1(x_j(0),0)\right) + O(h^2)\end{aligned}$$

since $\partial_x F_1(x_j(0),0)=0$. Note that

$$\partial_x^2 F_1(x_j(0),0)=\tfrac{1}{\beta}-\tfrac{1}{(\cosh x_0(\beta))^2}=H(\beta)>0,\quad \partial_h\partial_x F_1(x_j(0),0)=-\frac{1}{\beta}$$

are independent of j then

$$\delta_1=\delta_2=\delta=\tfrac{1}{\beta H(\beta)}>0.$$

Inserting these results in the expression for F_1 and expanding around $h=0$ we obtain

$$\begin{aligned}F_1(x_j(h),h)&=F_1(x_j(0),0)+\partial_x F_1(x_j(0),0)\delta h+\partial_h F_1(x_j(0),0)h+O(h^2)\\&=F_1(x_j(0),0)-\frac{x_j(0)}{\beta}h+O(h^2)=F_m-\sigma_j\frac{x_0(\beta)}{\beta}h+O(h^2)\end{aligned}$$

Then

$$F_1(x_1(h),h)-F_1(x_2(h),h)=\frac{2hx_0(\beta)}{\beta}>0,\quad \text{since } h>0,$$

and F_1 has a *global minimum* at $x_2(h)$. As in the case $\beta<1$ we extract the contribution of the minimum

$$\mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda}S_j\right]=\tfrac{x_2(h)-h}{\beta}+R(\beta,h,|\Lambda|)$$

where

$$|R(\beta,h,|\Lambda|)|\leqslant\frac{\int_{\mathbb{R}}dx\ |x-x_2|e^{-|\Lambda|[F_1(x,h)-F_m]}}{\int_{\mathbb{R}}dx\ e^{-|\Lambda|[F_1(x,h)-F_m]}}=\frac{N}{D}.$$

To estimate the integral in the numerator we distinguish three regions

$$I_1=\{x|\ |x-x_2(h)|<\epsilon\},\quad I_2=\{x|\ |x|>M\},\quad I_3=\{x|\ |x|\leqslant M,|x-x_2(h)|\geqslant\epsilon\}$$

where ϵ and are chosen in order to have $I_2\cap I_1=\varnothing$,

$$\partial_x^2 F_1(x,h)>c_1>0\quad\forall x\in I_1,\quad\text{and}\quad [F_1(x,h)-F_m]\geqslant\frac{c_2}{2}x^2\ \forall x\in I_3,$$

for some constant c_1,c_2. It is not difficult to see that such regions exist for the function F_1. Then

$$\int_{I_1}dx\ |x-x_2|e^{-|\Lambda|[F_1(x,h)-F_m]}\leqslant\int_{I_1}dx\ |x-x_2|e^{-\frac{|\Lambda|c_1}{2}(x-x_2)^2}\leqslant\int_{\mathbb{R}}dx\ |x-x_2|e^{-\frac{|\Lambda|c_1}{2}(x-x_2)^2}=\tfrac{2}{|\Lambda|c_1}$$

$$\begin{aligned}\int_{I_2}dx\ |x-x_2|e^{-|\Lambda|[F_1(x,h)-F_m]}&\leqslant\int_{I_2}dx\ |x-x_2|e^{-\frac{|\Lambda|c_2}{2}x^2}\\&\leqslant e^{-\frac{|\Lambda|c_2M^2}{4}}\int_{\mathbb{R}}dx\ |x-x_2|e^{-\frac{|\Lambda|c_2}{4}x^2}=e^{-\frac{|\Lambda|c_2M^2}{4}}O\left(\tfrac{1}{\sqrt{|\Lambda|}}\right)\end{aligned}$$

$$\int_{I_3}dx\ |x-x_2|e^{-|\Lambda|[F_1(x,h)-F_m]}\leqslant 2M\sup_{x\in I_3}\left[|x-x_2|e^{-|\Lambda|[F_1(x,h)-F_m]}\right]\leqslant e^{-|\Lambda|c(h,\epsilon,M)}.$$

In the third line we used $|x - x_2(h)| \geqslant \epsilon > 0 \ \forall x \in I_3$ and for some constant $\epsilon > 0$, since $F(x, h)$ is at a finite distance from the minimum. Putting all these bounds together we obtain an upper bound for the numerator

$$N = O\left(\tfrac{1}{|\Lambda|}\right).$$

To estimate the denominator note that

$$\partial_x^2 F_1(x, h) \leqslant \frac{1}{\beta} \quad \forall x, h$$

then

$$\int_{\mathbb{R}} dx\ e^{-|\Lambda|[F_1(x,h)-F_m]} \geqslant \int_{\mathbb{R}} dx\ e^{-\frac{|\Lambda|}{2\beta}(x-x_2)^2} = \sqrt{\tfrac{2\pi\beta}{|\Lambda|}}$$

hence

$$|R(\beta, h, |\Lambda|)| \leqslant \sqrt{\tfrac{|\Lambda|}{2\pi\beta}} O\left(\tfrac{1}{|\Lambda|}\right) \to_{|\Lambda|\to\infty} 0$$

Finally

$$\lim_{h\to 0+} \lim_{\Lambda\to\mathbb{Z}^d} \mathbb{E}_\Lambda^{\beta,h}\left[\tfrac{1}{|\Lambda|}\sum_{j\in\Lambda} S_j\right] = \lim_{h\to 0+} \tfrac{x_2(\beta,h)-h}{\beta} = \tfrac{x_0(\beta)}{\beta} > 0.$$

This concludes the proof. □