
Problem 1 (Conservation laws: shock, 1+3 points).
Consider
$$\begin{cases} \partial_t u + \partial_x F(u) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$
with
$$F(u) = u^2 + u, \qquad g(x) = \begin{cases} 1 & \text{for } x < 0, \\ -3 & \text{for } x > 0. \end{cases}$$

(i) Sketch the characteristics.

(ii) Construct an integral solution in $\mathbb{R} \times (0, \infty)$.

Problem 2 (Blow up in Burgers' equation, 1+3+3 points).
Consider
$$\begin{cases} \partial_t u + u \partial_x u = 0, \\ u(0, x) = g(x). \end{cases} \tag{1}$$
Let $g \in C^\infty(\mathbb{R})$ be such that g' is bounded and has a unique minimizer x_0 with $g'(x_0) < 0$, $g''(x_0) = 0$, $g'''(x_0) > 0$.

(i) Determine the largest $0 < t^* < \infty$ such that (1) is solved in $(0, t^*) \times \mathbb{R}$ by $u(t, x) = g(X(t, x))$, with $X(t, x)$ smooth function. Note that t^* depends on some X^*.

(ii) What happens with $\partial_t u, \partial_x u$ and the characteristic curves of (1) as $t \to t^*$?

(iii) Perform a local analysis (Taylor expansion, term comparison) to argue that
$$u(t^*, x) - g(X^*) \sim -(x - x^*)^{\frac{1}{3}}$$
for x close to x^*, where $x^* = X^* + g(X^*)t^*$. Here $a \sim b$ means $a = Cb$ for some constant $C > 0$, up to higher order terms.

Problem 3 (Unbounded entropy solution to Burgers' equation, 1+2+1 points).
Our goal is to study entropy solutions in $L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R})$ instead of $L^\infty(\mathbb{R}_+ \times \mathbb{R})$.

(i) Show that

$$u(x,t) := \begin{cases} -\frac{2}{3}\left(t + \sqrt{3x + t^2}\right) & \text{if } 4x + t^2 > 0, \\ 0 & \text{if } 4x + t^2 < 0 \end{cases}$$

is an (unbounded) entropy solution to Burgers' equation $\partial_t u + \partial_x(\frac{u^2}{2}) = 0$.

(ii) Verify the Rankine-Hugoniot condition.

(iii) Sketch the projected characteristics.

Problem 4 (Shock curves, 1+4 points).
Consider

$$\begin{cases} \partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with

$$g(x) = \begin{cases} 1 & \text{for } x < 0, \\ -1 & \text{for } 0 < x < 1, \\ 0 & \text{for } x > 1. \end{cases}$$

(i) Sketch the characteristics diagram.

(ii) Construct an entropy solution and sketch the solution including all shock curves.

Total: 20 points