Nonlinear Partial Differential Equations II University of Bonn
Summer term 2017 Prof. Dr. M. Disertori
Problem Sheet 8 (due Wednesday 28.06.2017) L. Borasi, M. Lager

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Problem 1 (Viscous Burgers' equation, 2+2 points).
Consider the viscous Burgers' equation for $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$

$$\partial_t u + u\partial_x u = \nu \partial_x^2 u,$$

with $\nu \in \mathbb{R}$.

(i) Identify the exponents n, m such that self-similar solutions of the form $u(t,x) = t^m f(xt^n)$ can be obtained. Write down the resulting ODE for the function f.

(ii) Let $v(t,x)$ be a solution to the heat equation $\partial_t v = \nu \partial_x^2 v$. Show that, if we make the transformation $u(t,x) = -2\nu \frac{1}{v} \partial_x v$, we obtain a solution to the viscous Burgers' equation.

Problem 2 (Kortweg-de Vries equation, 2 points).
Consider the Korteweg-de Vries (KdV) equation for $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$

$$\partial_t u + 6u\partial_x u + \partial_x^3 u = 0.$$

We look for a plane wave solution $u(t,x) = f(x - ct)$. Write the resulting ODE for f. Assuming that both f and its derivatives vanish at infinity, integrate this equation once to get a second order ODE for f.

Problem 3 (Method of characteristics, 4 points).
Use the method of characteristics to solve the following equation for $u : \mathbb{R}^2 \to \mathbb{R}$

$$\begin{cases} u\partial_{x_1} u + \partial_{x_2} u = 1 & \text{in } \mathbb{R}^2, \\ u(x_1, x_2) = \frac{1}{2} x_1 & \text{on } \Gamma = \{x_2 = x_1\}. \end{cases}$$

Hint: A reasonable condition on $\Phi : (s, X) \to (x_1, x_2)$ is to impose $x_1(0,x) = x_2(0,x) = X$.

Problem 4 (A quasilinear PDE, 4 points).
Consider the quasilinear initial value problem for $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$

$$\begin{cases} \partial_t u + b(u)\partial_x u = 0 \\ u(0,x) = g(x) \end{cases} \tag{1}$$

with $b \in C^1(\mathbb{R})$, $g \in C^1(\mathbb{R})$.

(i) Use the method of characteristic to show that a C^1 solution of (1) must satisfy the implicit relation

$$u(t,x) = g(x - tb(u(t,x))).$$

(ii) Show that this solution is defined for all $t > 0$ if and only if $b \circ g$ is a non decreasing function.

Problem 5 (Linear transport, 3+3 points).
Consider the linear transport equation for $u, b : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$

$$\partial_t u(t,x) + b(t,x)\partial_x u(t,x) = 0.$$

(i) Determine the characteristic curves $x = x(t)$, $x(0) = x_0$ for the transport velocities

  (a) $b(t,x) = x$,
  (b) $b(t,x) = \omega \cos(\omega t + \phi)$, $\omega, \phi \in \mathbb{R}$,
  (c) $b(t,x) = g(x_0)$, where

$$g(x_0) = \begin{cases} 0 & \text{if } x_0 \leq 0, \\ -x_0 & \text{if } 0 < x_0 < 1, \\ -1 & \text{if } x_0 \geq 1. \end{cases}$$

In each case, sketch the trajectories $x(t)$ for $t > 0$ and several values of x_0.

(ii) Give a simple yet nontrivial condition on $b(t,x)$ such that

  (a) characteristic curves do not intersect,
  (b) a solution $x(t)$ exists for all $t > 0$.

Total: 20 points