Nonlinear Partial Differential Equations II University of Bonn
Summer term 2017 Prof. Dr. M. Disertori
Problem Sheet 7 (due Wednesday 21.06.2017) L. Borasi, M. Lager

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Problem 1 (Telegraph equation, 4 points).
Show there exists at most one smooth solution of this initial/boundary-value problem for the telegraph equation

$$\begin{aligned} \partial_t^2 u + K\partial_t u - \Delta u &= f && \text{in } (0,1)\times(0,T) \\ u &= 0 && \text{on } (\{0\}\times[0,T])\cup(\{1\}\times[0,T]) \\ u = g, \partial_t u &= h && \text{on } (0,1)\times\{0\}, \end{aligned}$$

where $K \in \mathbb{R}^+$, $f : (0,1)\times(0,T) \to \mathbb{R}$ and $g, h : (0,1) \to \mathbb{R}$.
Hint: Consider the energy $E(t) = \frac{1}{2}(\|\partial_t u\|^2_{L^2((0,1))} + \|Du\|^2_{L^2((0,1))})$.

Problem 2 (2+2+2+2* points).
Let $R \subset \mathbb{R}^d$ be a rectangle $R = (0, r_1) \times \cdots \times (0, r_d)$. Let $g \in H^1_0(R)$ and $h \in L^2(R)$. Consider the homogeneous wave equation

$$\begin{aligned} \partial_t^2 u - \Delta u &= 0 && \text{in } R\times(0,T) \\ u &= 0 && \text{on } \partial R\times(0,T) \\ u = g, \partial_t u &= h && \text{on } R\times\{0\}. \end{aligned} \tag{1}$$

(i) Let $\{w_k\} \subset H^1_0(R)$ be an orthonormal basis of $L^2(R)$ that consists of eigenfunctions of $-\Delta$, such that $-\Delta w_k = \lambda_k w_k$ with $0 < \lambda_1 \le \lambda_2 \le \dots$ and $\lambda_k \to \infty$ as $k \to \infty$. Show that the weak solution u of (1) has a representation of the form

$$u(t,x) = \sum_{k\in\mathbb{N}} c_k(t) w_k(x),$$

i.e., determine the $c_k(t)$ and show that the series converges in an appropriate sense.

(ii) Determine an orthonormal basis $\{w_k : k \in \mathbb{N}\} \subset H^1_0(R)$ of $L^2(R)$ of eigenfunctions of $-\Delta$.

Hint: Consider $d = 1$ first and use separation of variables $w(x_1, \dots, x_d) = w_1(x_1)\cdots w_d(x_d)$ for $d > 1$.

(iii) Denote by $N_R(m)$ the number of eigenvalues λ_k such that $\lambda_k \le m$. For $d = 2$ show Weyl's formula

$$|R| = (2\pi)^d \omega_d^{-1} \lim_{m\to\infty} \frac{N_R(m)}{m^{d/2}},$$

where ω_d is the volume of the unit ball in $\mathbb{R}^d$.

(iv*) Can one hear the size of a rectangular drum membrane?

Hint: Think about how the lowest eigenvalue λ_1 depends on R.

Problem 3 (2 points).
Let $p, q : \mathbb{R} \to \mathbb{R}$ be C^1 functions. Show that the system

$$\begin{cases} \partial_t u_1 - \partial_x q(u_2) = 0 \\ \partial_t u_2 - \partial_x p(u_1) = 0 \end{cases}$$

is strictly hyperbolic if $p', q' > 0$.

Problem 4 (Shallow water equations, 2+2 points).
Consider the one dimensional shallow water equations

$$\begin{cases} \partial_t h + \partial_x (vh) = 0 \\ \partial_t (vh) + \partial_x (v^2 h + h^2/2) = 0, \end{cases} \tag{2}$$

where $h, v \in C^1(\mathbb{R}_+ \times \mathbb{R})$ are the unknown functions.

(i) Verify that (2) forms a strictly hyperbolic system, provided $h > 0$.

Hint: Consider $q = vh$ instead of v.

(ii) Show that for a smooth solution (h, v), with $h > 0$, (2) can be recast into this alternative conservation law form:

$$\begin{cases} \partial_t h + \partial_x (vh) = 0 \\ \partial_t v + \partial_x (v^2/2 + h) = 0. \end{cases}$$

Check that this is a strictly hyperbolic system (provided $h > 0$).

Problem 5 (4 points).
Define for $z \in \mathbb{R}$, $z \neq 0$, the matrix function

$$\mathbf{B}(z) := e^{-\frac{1}{z^2}} \begin{pmatrix} \cos(2/z) & \sin(2/z) \\ \sin(2/z) & -\cos(2/z) \end{pmatrix},$$

and set $\mathbf{B}(0) = 0$. Show that $\mathbf{B}$ is C^∞ and has real eigenvalues, but we cannot find unit-length right eigenvectors $\{\mathbf{r}_1(z), \mathbf{r}_2(z)\}$ depending continuously on z near 0. What happens to the eigenspaces as $z \to 0$?

Total: 20 points