Nonlinear Partial Differential Equations II
Summer term 2017
Problem Sheet 6 (due Wednesday 14.06.2017)

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Problem 1 (Maximum principle, 2+3 points).
Let $u_0 \in C_c^\infty(B(0,2))$ with $\operatorname{supp} u_0 \subset (B(0,2) \backslash B(0,1))$ such that $u_0 \geq 0$ and $u_0 \not\equiv 0$. Let u be the solution of the heat equation $\partial_t u - \Delta u = 0$ in $(0,\infty) \times B(0,2)$ with $u = 0$ on $[0,\infty) \times \partial B(0,2)$ and $u(0,x) = u_0(x)$.

(i) Show that $u(t,x) > 0$ for all $t > 0$ and all $x \in B(0,2)$.

(ii) Show that there exists $M > 0$ and $c > 0$ such that $\inf_{x \in B(0,1)} u(t,x) \leq M e^{-\frac{c}{t}}$.

Hint: Extend u_0 by zero to $\mathbb{R}^d$ and let $\tilde{u}$ be the solution of the heat equation in $[0,\infty) \times \mathbb{R}^d$ with initial datum u_0 and use the maximum principle to compare u and $\tilde{u}$.

Problem 2 (2+3 points).
Suppose $u, v \in C_1^2(U_T) \cap C(\bar{U}_T)$ with $u \geq v$ on Γ_T.

(i) Let $K \in R$. Show that either $u \geq v$ for every $(t,x) \in \bar{U}_T$ or there exists $(t_0, x_0) \in U_T$ depending on K with the following properties:

$$\begin{aligned}
u(t_0,x_0) &< v(t_0,x_0) \\
Du(t_0,x_0) &= Dv(t_0,x_0) \\
\Delta u(t_0,x_0) &\geq \Delta v(t_0,x_0) \\
\partial_t u(t_0,x_0) &\leq \partial_t v(t_0,x_0) - K(v(t_0,x_0) - u(t_0,x_0)).
\end{aligned}$$

Hint: Show and use that without loss of generality you may assume $v = 0$ and $K = 0$.

(ii) Let $b_i : \mathbb{R} \to \mathbb{R}$ be bounded, let $F : \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous with Lipschitz constant k. Suppose that, in addition,

$$\begin{aligned}
u_t &\geq \Delta u + \sum_{i=1}^d b_i \partial_i u + F(u(t,x)), \\
v_t &\leq \Delta v + \sum_{i=1}^d b_i \partial_i v + F(v(t,x)).
\end{aligned}$$

Show that $u \geq v$ in $\bar{U}_T$.

Problem 3 (5 points).
Let $U \subset \mathbb{R}^d$ be open and bounded with smooth boundary. Suppose $0 < t_1 < T$. Set $\omega_T := (t_1, T] \times U$ and $\Gamma_T = \partial\omega_T$, Suppose $u \in C^4(U_T) \cap C^2(\bar{U}_T)$ satisfies

$$\begin{aligned} u \geq \varepsilon > 0 \quad & \text{in } U_T \\ \partial_t u - \Delta u = 0 \quad & \text{in } U_T \\ u\Delta u - |Du|^2 \geq -\frac{d}{2t}u^2 \quad & \text{on } \Gamma_T. \end{aligned}$$

Show that

$$\frac{\partial_t u}{u} - \frac{|Du|^2}{u^2} + \frac{d}{2t} \geq 0 \quad \text{in } \omega_T.$$

Hint: Show that there is a nonlinear heat equation for which $\Delta \log u$ is a supersolution and $-\frac{d}{2t}$ is a subsolution. Then apply Problem 2.

Problem 4 (5 points).
Let U be a bounded domain of $\mathbb{R}^d$ with smooth boundary ∂U. We consider on U the linear, (uniformly) elliptic operator of the form

$$Lu(x) = -\sum_{i,j=1}^{d} a^{ij}(x)D_iD_ju(x) + \sum_{k=1}^{d} b_k(x)D_ku(x) + c(x)u(x) \tag{1}$$

with coefficients $a^{ij}, b_k, c \in C^\infty(\bar{U})$, with $a_{ij} = a_{ji}$ satisfying $a(x) \geq \theta\,\mathrm{Id}$, $\theta > 0$, $\forall x$.

Show that there exist constants $\beta > 0$ and $\gamma \geq 0$ such that for all $u \in H_0^2(U)$ the following inequality holds

$$\beta\|u\|_{H^2(U)} \leq (Lu, -\Delta u)_{L^2(U)} + \gamma\|u\|^2_{L^2(U)}\,. \tag{2}$$

Total: 20 points