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**Problem 1 (A nonlinear parabolic equation II, 2 + 3 + 2 = 7 points).**
We continue with Problem 3 from Sheet 4. Let $d = 4$ and let $U \subset \mathbb{R}^d$ be open and bounded with smooth boundary. Let $0 < T < \infty$ and $I = (0, T)$. Set

$$\mathcal{V} := \{\phi \in C_c^\infty(U; \mathbb{R}^d) : \mathrm{div}\phi = 0\},$$

let $H$ be the closure of $\mathcal{V}$ in $L^2(U; \mathbb{R}^d)$, let $V$ be the closure of $\mathcal{V}$ in $H^1(U; \mathbb{R}^d)$, and let $W$ be the closure of $\mathcal{V}$ in $H^4(U; \mathbb{R}^d)$. Set

$$b(u, v, w) := \int_U \sum_{i,j=1}^d u_i(\partial_i v_j)w_j.$$

Let $f \in L^2(I; V')$, and $u_0 \in H$. Let $\{w_k\}$ be an orthonormal basis of $H$ and an orthogonal basis of $V$ such that the orthogonal projections $P_m : H \to \mathrm{span}\{w_1, \dots, w_m\}$ are bounded uniformly in $m$ as maps $H \to H$, $W \to W$ and $W' \to W'$. Let

$$u_m(t) := \sum_{k=1}^m d_k^{(m)}(t)w_k$$

be a solution of

$$\begin{aligned} &d_k^{(m)}(0) = (u_0, w_k)_H, \\ &u_m'(t)[w_k] + (Du_m(t), Dw_k)_H + b(u_m(t), u_m(t), w_k) = f(t)[w_k] \\ &\text{a.e. } t \in I, \quad \forall 1 \le k \le m. \end{aligned} \tag{1}$$

(i) For $w \in W$ set $(B_m(t))(w) := b(u_m(t), u_m(t), w)$. Show that there exists a constant $0 < C < \infty$ depending only on $T, f$ and $u_0$ such that

$$\|B_m\|_{L^\infty(I;W')} \le C,$$

and deduce that there is a constant $0 < C' < \infty$ depending only on $T, f$ and $u_0$ such that

$$\|u_m'\|_{L^2(I;W')} \le C'.$$

*Hint:* Use Sobolev embedding, Problem 3 (ii) from Sheet 4 and uniform boundedness of $P_m$.

(ii) Show that there exists $u \in L^2(I; V) \cap L^\infty(I; H)$ with $u' \in L^2(I; W')$ such that

$$u'(t)[v] + (Du(t), Dv)_H + b(u(t), u(t), v) = f(t)[v] \quad \forall v \in V.$$

*Hint:* Use Problem 2 from Sheet 4, and pass to the limit in (1) for fixed $w_k$ first. Use weak convergence for the linear terms, and use appropriate continuity properties for the nonlinear term.

(iii) In which sense does one have

$$u(x,0) = u_0(x) \quad \text{for } x \in U?$$

*Hint:* You might, e.g., note that $u \in C^0(I; W')$.

**Problem 2 (Improved regularity, 2 + 2 + 2 + 1 = 7 points).**
Let $U \subset \mathbb{R}^d$ be open, bounded with smooth boundary. Let $0 < T < \infty$ and $I = (0,T)$. Let $L$ be an elliptic operator with coefficients $a_{ij}, b_i, c$ for $i,j = 1,\dots,d$ smooth on $\bar{U}$, independent of $t$ and $a_{ij} = a_{ji}$. Let $f \in L^2(I; L^2(U)), u_0 \in H_0^1(U)$. Suppose $u \in L^2(I; H_0^1(U))$ with $u' \in L^2(I; (H_0^1(U))')$ is the weak solution of

$$\begin{aligned} \partial_t u + Lu &= f && \text{in } U_T \\ u &= 0 && \text{on } \partial U \times [0,T] \\ u &= u_0 && \text{on } U \times \{t = 0\}. \end{aligned}$$

Let $u_m(t) := \sum_{k=1}^m d_k^{(m)}(t) w_k$ the Galerkin approximation for $u$ (see lecture).

(i) Show that $u_m, u_m' \in L^2(I, C^\infty(\bar{U}))$ and $u_m'(t)[v] = (u_m'(t), v)_{L^2(U)}$ for all $v \in H_0^1(U)$. Moreover let $A : H_0^1 \times H_0^1 \to \mathbb{R}$ be defined by

$$A[u,v] := \int_u \sum_{i,j=1}^d a_{ij} \partial_i u \partial_j v dx.$$

Prove that $t \mapsto A[u_m(t), u_m(t)]$ is absolutely continuous with weak derivative $2A[u_m(t), u_m'(t)]$.

(ii) Show that

$$\int_0^T \|u_m'\|_{L^2(U)}^2 dt + \sup_{0 \le t \le T} \|u_m(t)\|_{H_0^1(U)}^2 \le C(\|u_0\|_{H_0^1(U)}^2 + \|f\|_{L^2(I;L^2(U))}^2).$$

*Hint:* Multiply the weak formulation for $u_m$ by $d_k^{(m)'}(t)$ and sum over $k = 1,\dots,m$ and use Young's inequality on the different terms.

(iii) Show that $u \in L^\infty(I; H_0^1(U))$, $u' \in L^2(I; L^2(U))$ with

$$\|u\|_{L^\infty(I;H_0^1(U))} + \|u'\|_{L^2(I;L^2(U))} \le C(\|u_0\|_{H_0^1(U)} + \|f\|_{L^2(I;L^2(U))}).$$

*Hint:* Pass to limits $m = m_l \to \infty$ and note that $\int_a^b (v, u_m(t))_{H_0^1(U)} dt \le C\|v\|_{H_0^1(U)} |b-a|$ for $0 \le a < b \le T$ and $v \in H_0^1(U)$.

(iv) Prove that $u \in L^2(I; H^2(U))$ and

$$\|u\|_{L^2(I;H^2(U))} \le C(\|u_0\|_{H_0^1(U)} + \|f\|_{L^2(I;L^2(U))}).$$

*Hint:* You may use without prove the elliptic regularity theorem (Theorem 4 in §6.3.2 in Evans).

**Problem 3 (Energy loss in parabolic equations, 2 + 2 + 2 = 6 points).**

(i) Let $d \geq 2$. Show that there exists a constant $C_d$ which only depends on $d$ such that for all $f \in L^1(\mathbb{R}^d)$ with $Df \in L^2(\mathbb{R}^d)$

$$\|f\|_{L^2(\mathbb{R}^d)} \leq C_d \|f\|_{L^1(\mathbb{R}^d)}^{2/(d+2)} \|Df\|_{L^2(\mathbb{R}^d)}^{d/(d+2)}$$

*Hint:* For $d \geq 3$ use the critical Sobolev embedding and Hölder's inequality. For $d = 2$ show first $\|f\|_{L^4} \leq C\|f\|_{L^2}^{1/2}\|Df\|_{L_2}^{1/2}$ by applying the Sobolev embedding for $f^2$ and then use Hölder's inequality.

(ii) Let $a_{ij} : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be smooth and bounded with bounded derivatives and let $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be a smooth solution of

$$\partial_t u - \sum_{i,j=1}^{d} \partial_i(a_{ij}(t,x)\partial_j u) = 0, \quad \text{where } \sum_{i,j} a_{ij}(t,x)\xi_i\xi_j \geq \theta|\xi|^2, \quad \theta > 0.$$

Assume also that $u \geq 0, Du \in L^\infty((0,\infty), L^2(\mathbb{R}^d)), u \in L^\infty((0,\infty), L^1(\mathbb{R}^d))$. Let

$$m(t) := \int_{\mathbb{R}^d} u(t,x)dx, \quad E(t) := \int_{\mathbb{R}^d} u^2(t,x)dx.$$

Show that

$$m \equiv m_0, \quad E'(t) \leq -C\theta m^{-4/d} E^{(d+2)/d}.$$

*Hint:* Let $\chi \in C_c^\infty(B(0,1)), \chi_R(x) = \chi(x/R)$ and compute first $\frac{d}{dt}\int_{\mathbb{R}^d} \chi_R^2 v dx$ with $v(x) = u(t,x)$ or $v(x) = u^2(t,x)$. Then take $R \to \infty$. For the estimate for $m$ you may use that the $a_{ij}$ are smooth.

(iii) Conclude that $E(t) \leq Cm_0^2\theta^{-d/2}t^{-d/2}$.

*Hint:* Consider $\frac{d}{dt}E^{-2/d}$.

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Total: 20 points