
Problem 1 (Example for an evolution triple, 1+1+1=3 points).
Let

$$H := \ell^2 = \left\{ u = (u_n)_{n\geq 1} \;\middle|\; \sum_{n=1}^{\infty} u_n^2 < \infty \right\}$$

equipped with the scalar product $(u,v)_H = \sum_{n=1}^{\infty} u_n v_n$. Let

$$V := \left\{ u = (u_n)_{n\geq 1} \;\middle|\; \sum_{n=1}^{\infty} n^2 u_n^2 < \infty \right\}$$

equipped with the scalar product $(u,v)_V = \sum_{n=1}^{\infty} n^2 u_n v_n$. Moreover let

$$A := \left\{ f = (f_n)_{n\geq 1} \;\middle|\; \sum_{n=1}^{\infty} \frac{1}{n^2} f_n^2 < \infty \right\}$$

equipped with the scalar product $(u,v)_A = \sum_{n=1}^{\infty} n^{-2} u_n v_n$.

(i) Let $i_1 : V \to H$, $i_1(u) = u$ and $i_2 : V \to H$, $i_2(u)_n = n u_n$. Show that i_1 and i_2 are both continuous embeddings and $i_q(V)$ is dense in H for $q = 1, 2$.

(ii) Let $\Phi_H : H \to H'$ be the canonical isometric isomorphism and $J_q : H' \to V'$ $J_q(L) = L \circ i_q$ for $q = 1, 2$. Determine the explicit form of $\mathcal{J}_q = J_q \circ \Phi_H \circ i_q : V \to V'$. Show that $\mathcal{J}_2$ is an isometric isomorphism but $\mathcal{J}_1$ is not.

(iii) Show that there exists an extension $E\mathcal{J}_1 : A \to V'$ of $\mathcal{J}_1$ which is an isometric isomorphism.

Problem 2 (Aubin-Lions-Simon, 2+8=10 points).
Let $X \subset Y \subset Z$ be Banach spaces and assume that $X \hookrightarrow Y$ is compact and $Y \hookrightarrow Z$ is continuous.

a) Show that for every $\delta > 0$ there exists $c_\delta > 0$ such that

$$\|x\|_Y \leq \delta \|x\|_X + c_\delta \|x\|_Z \quad \text{for all } x \in X. \tag{1}$$

Hint: Argue by contradiction and consider the sequence x_n such that

$$\|x_n\|_Y > \delta_0 \|x_n\|_X + n \|x_n\|_Z$$

for $n \geq 1$.

b) Let $I = (0, T)$ and let for $1 \leq p < \infty$

$$A_p := \left\{ u \in L^p(I; X), u' \in L^1(I; Z) \right\}.$$

Our goal is to prove that the embedding $A_p \hookrightarrow L^p(I; Y)$ is compact.

(i) Using (1) prove that if $(v_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^p(I; Z)$ and is bounded in $L^p(I; X)$ then it is also a Cauchy sequence in $L^p(I; Y)$.

Let $(u_n)_n$ be a sequence such that: 1. $(u_n)_n$ is a bounded sequence in $L^p(I; X)$; 2. $(u_n')_n$ is a bounded sequence in $L^1(I; Z)$. Let $\theta \in C^\infty([0, T]; \mathbb{R})$ with $\theta(T) = 0$ and $\theta(0) = 1$. Let

$$u_n = \theta u_n + (1 - \theta) u_n =: v_n + w_n.$$

Extend $v_n(t)$ to $\mathbb{R}_+$ setting $v_n(t) = 0$ for all $t > T$. Taking $h > 0$ fixed, split $v_n(t)$ as follows

$$v_n(t) = \frac{1}{h} \int_t^{t+h} v_n(s) ds + \frac{1}{h} \int_t^{t+h} (v_n(t) - v_n(s)) ds =: a_{n,h}(t) + b_{n,h}(t).$$

(ii) Show that $\sup_{t>0} \|a_{n,h}(t)\|_X \leq C_1(h)$ for some constant $C_1(h)$ which depends only on h. Hence $t \to a_{n,h}(t)$ takes values, independently of n, in a bounded subset of X and therefore in a compact subset of Z.

(iii) Show that $\sup_{t>0} \|a'_{n,h}(t)\|_Z \leq C_2(h)$ for some constant $C_2(h)$ which depends only on h.

(iv) Show that $\|b_{n,h}\|_{L^p(I;Z)} \leq C h^{1/p}$, for some constant $C > 0$ (Hint: use Hölder inequality and Fubini's theorem).

(v) For $k \geq 1$ set $h_k = 1/k$. Use Arzelà-Ascoli to extract from a_{n,h_k} a subsequence $a_{\psi(k),h_k}$ convergent in $L^p(I; Z)$ (Hint: start with h_1 fixed).

(vi) Use (iv) and (v) to show that we can extract from $v_n = a_{n,h} + b_{n,h}$ a subsequence which is Cauchy in $L^p(I; Z)$.

(vii) Note that a similar argument holds for w_n. Conclude that $A_p \hookrightarrow L^p(I; Y)$ is compact.

Problem 3 (A nonlinear parabolic equation I, 3+2+2*+2 = 7+2* points).
We want to study the nonlinear equation

$$\partial_t u - \Delta u + uDu = f. \tag{2}$$

Let $d = 4$ and let $U \subset \mathbb{R}^d$ be open and bounded with smooth boundary. Let $0 < T < \infty$ and $I = (0, T)$. Set

$$\mathcal{V} := \{\phi \in C_c^\infty(U; \mathbb{R}^d) : \text{div}\phi = 0\},$$

let H be the closure of $\mathcal{V}$ in $L^2(U; R^d)$, let V be the closure of $\mathcal{V}$ in $H^1(U; \mathbb{R}^d)$, i.e.,

$$V = \{\phi \in H_0^1(U; \mathbb{R}^d) : \text{div}\phi = 0\}.$$

Set

$$b(u, v, w) := \int_U \sum_{i,j=1}^d u_i(\partial_i v_j) w_j.$$

(i) Show that for all $u, v, w \in V$,

$$b(u, v, w) \leq C\|u\|_V \|v\|_V \|w\|_V, \tag{3}$$

$$b(u, u, v) = -b(u, v, u). \tag{4}$$

Hint: For (4) assume first that $u, v \in \mathcal{V}$, integrate by parts and then use (3).

(ii) Let $f \in L^2(I; V')$, and $u_0 \in H$. Derive a reasonable notion of weak solutions to (2) for $u \in L^2(I; V)$.

(iii*) Let $\{w_k\}$ be an orthonormal basis of H and an orthogonal basis of V. Show that there exists a $T > 0$ such that for all $m \in \mathbb{N}$ there exists a $u_m(t)$ of the form

$$u_m(t) := \sum_{k=1}^m d_k^{(m)}(t) w_k$$

solving

$$\begin{aligned} &(u_m(0), w_k)_H = (u_0, w_k)_H, \\ &u_m'(t)[w_k] + (Du_m(t), Dw_k)_{L^2(U)} + b(u_m(t), u_m(t), w_k) = f(t)[w_k] \\ &\text{a.e. } t \in I, \quad \forall 1 \leq k \leq m. \end{aligned} \tag{5}$$

(iv) Show that there exists a constant $0 < C < \infty$ depending only on T, f and u_0 such that

$$\|u_m\|_{L^2(I;V)} + \|u_m\|_{L^\infty(I;H)} \leq C.$$

Hint: Multiply (5) by $d_k^{(m)}(t)$ and sum over k.

Total: 20 points