
Let X be a Banach space and $I = (0, T)$ for $0 < T < \infty$.

Problem 1 (Density of simple functions in $L^p(I; X)$, 3+3+3* points).

(i) Prove that the family of integrable simple functions $s : I \to X$ is dense in $L^p(I; X)$ for $1 \le p < \infty$.

Hint: Find for every $u \in L^p(I; X)$ a suitable sequence of simple functions approximating u pointwise a.e. and conclude with dominated convergence.

(ii) Let $u \in L^\infty(I; X)$. Assume that there exists a sequence $\{s_n\}_{n\in\mathbb{N}}$ of simple functions, such that $\lim_{n\to\infty} \|s_n - u\|_{L^\infty(I;X)} = 0$. Let $\tilde{I} \subset I$ be the maximal set such that $|I \setminus \tilde{I}| = 0$ and $\|s_n - u\|_{L^\infty(I;X)} = \sup_{t\in\tilde{I}} \|s_n(t) - u(t)\|_X$ for all $n \in \mathbb{N}$. Show that $\overline{u(\tilde{I})}$ is compact.

(iii*) *Bonus problem:* Find a $u \in L^\infty(I; X)$ such that $\overline{u(\tilde{I})}$ is not compact for all $\tilde{I} \subset I$ such that $|I \setminus \tilde{I}| = 0$.

Problem 2 (Extension of Bochner functions, 1+2 points).
Let H be a Hilbert space and $X \hookrightarrow H \cong H' \hookrightarrow X'$ an evolution triple. Let $u \in L^2(I; X)$ with $u' \in L^2(I; X')$. For $0 < \delta < T$ define the extension $u_e : \mathbb{R} \to X$ in the following way. Extend u by reflection at 0 and T to $u^{ext} : (-\delta, T + \delta) \to X$ i.e.

$$u^{ext}(t) = \begin{cases} u(-t) & \text{for } -\delta < t < 0, \\ u(t) & \text{for } t \in [0, T], \\ u(2T - t) & \text{for } T < t < T + \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi \ge 0, \phi = 1$ in I and $\operatorname{supp} \phi \subset (-\delta, T + \delta)$. Set $u_e = \phi u^{ext}$. Prove that

(i) $u_e \in L^2((-\delta, T + \delta); X)$ with $u = u_e$ in I and

(ii) u_e is weakly differentiable with $u_e' \in L^2((-\delta, T + \delta); X')$ with $u_e' = u'$ in I.

Problem 3 (Lemma 2.14, 1+3 points).
Let $U \subset \mathbb{R}^d$ open bounded and $X \hookrightarrow H \cong H' \hookrightarrow X'$ our usual evolution triple with $X = H_0^1(U)$, $H = L^2(U)$.

Let $\{w_k\}_{k\in\mathbb{N}}$ be smooth functions such that $\{w_k\}_{k\in\mathbb{N}}$ is an orthonormal basis of H and and orthogonal basis of X. Let $V_k = \operatorname{span}\{w_1, \dots, w_k\}$ and consider the new evolution triple $X_k \hookrightarrow H_k \cong H_k' \hookrightarrow X_k'$ with $X_k = (V_k, (\,,\,)_X)$ and $H_k = (V_k, (\,,\,)_H)$.

Let i_k and J_k be the natural embeddings:

$$\begin{aligned} i_k : X_k &\to X & \qquad J_k : X' &\to X_k' \\ x &\mapsto i_k(x) = x & L &\mapsto J_k(L) = L \circ i_k = L_{|X_k}. \end{aligned}$$

For every $h_k \in X_k$ let $L_{h_k} \in X_k'$ given by $L_{h_k}(v) = (h_k, v)_X$ for all $v \in X_k$. Define the extension

$$\begin{aligned} E_k : X_k' &\to X' \\ L_{h_k} &\mapsto E_k(L_{h_k}) \text{ with } E_k(L_{h_k})(v) = (h_k, v)_X \text{ for all } v \in X. \end{aligned}$$

(i) Let $u \in L^2(I; X)$ with $u' \in L^2(I; X')$. Assume $u(I) \subset X_k$. Prove that $u \in L^2(I; X_k)$ and $u' \in L^2(I; X_k')$.

(ii) Let $u \in L^2(I; X_k)$ with $u' \in L^2(I; X_k')$. Prove that $i_k \circ u \in L^2(I; X)$, $(i_k \circ u)' \in L^2(I; X_k')$ and $(i_k \circ u)' = E_k(u')$.

Problem 4 (Properties of evolution triples, 2+3+2 points).
Let H be a Hilbert space. Consider the evolution triple $X \hookrightarrow H \cong H' \hookrightarrow X'$. Let $T : H \to X'$ be the continuous embedding from H to X'.

(i) Given $f \in X'$, show that $f \in \operatorname{Range}(T)$ if and only if there exists a constant $a \geq 0$ such that $|f(x)| \leq a\|x\|_H$, for all $x \in X$.

(ii) Let X be reflexive. Show that $T(H)$ is dense in X'.

Hint: You may use without prove the following: Let X be a normed space, $Y \subset X$ be a closed subspace and $x_0 \in X \setminus Y$. Then there exists $x' \in X'$ such that

$$x' = 0 \text{ on } Y, \quad \|x'\|_{X'} = 1, \quad x'(x_0) = \operatorname{dist}(x_0, Y).$$

(iii) Let X be reflexive. Show that for every $f \in L^2((0,T); X')$ there exists a sequence $(f_n)_{n\in\mathbb{N}} \subset L^2((0,T); H)$ such that $T(f_n) \to f$ in $L^2((0,T); X')$.

Hint: Approximate f by simple functions.

Total: 20 points