Nonlinear Partial Differential Equations II | University of Bonn
Summer term 2017 | Prof. Dr. M. Disertori
Problem Sheet 2 (due Wednesday 10.05.2017) | L. Borasi, M. Lager

---

Problem 1 (Counterexample, 1+1 points).
Recall that for *real-valued* functions *absolute continuity* is equivalent to weak differentiability with integrable derivative. This statement is false for *Banach space* valued functions.

Let $u : (0,1) \to L^1(0,1)$ by
$$u(t)(x) = t\chi_{[0,t]}(x)\,.$$

(i) Show that u is Lipschitz continuous and therefore absolutely continuous.

(ii) Show that its weak derivative u' does not exist.

Problem 2 (2+2 points).
Let $U \subset \mathbb{R}^d$ open and bounded, $I = (0,T)$ and $U_T = U \times (0,T] \subset \mathbb{R}^{d+1}$.

(i) Prove that $C(\bar{U}_T)$ is isometrically isomorphic to $C([0,T]; C(\bar{U}))$, which is the space of all continuous functions $u : [0,T] \to C(\bar{U})$.

(ii) Let $u \in C(\bar{U}_T)$. Prove that u is differentiable in t with $\partial_t u \in C(\bar{U}_T)$ if and only if $u \in C^1([0,T]; C(\bar{U}))$, i.e., $u \in C([0,T]; C(\bar{U}))$ is Fréchet-differentiable with continuous derivative $\frac{d}{dt}u : I \to C(\bar{U})$.

Hint: Use the fundamental theorem of calculus.

Problem 3 (Properties of the mollifiers, 1+1+1+1+1=5 points).
Let $\eta_\epsilon \in C_c^\infty(\mathbb{R})$ be a standard mollifier (that is, $\eta_\epsilon(t) = \frac{1}{\epsilon}\eta(t/\epsilon)$ with $\eta \in C_c^\infty(\mathbb{R})$ symmetric, $0 \le \eta \le 1$, $\int_{\mathbb{R}} \eta(t)dt = 1$, and $\epsilon > 0$).

Let $u \in L^1_{\text{loc}}(I; X)$, where $I = (0,T)$. Define $u^\epsilon(t) := (\eta_\epsilon * u)(t) = \int_0^T \eta_\epsilon(t-s)u(s)ds$, where the integral is in the Bochner sense.

(i) Show that $u^\epsilon \in C^\infty((\epsilon, T-\epsilon); X)$ for all $\epsilon > 0$.

(ii) Show that for almost every $t \in I$ we have $\lim_{\epsilon\downarrow 0} u_\epsilon(t) = u(t)$.

(iii) Show that if $u \in C(I;X)$ then, for $\epsilon \downarrow 0$, $u^\epsilon \to u$ uniformly on compact subsets of I.

(iv) Let $1 \le p < \infty$ and $u \in L^p(I;X)$, show that, for $\epsilon \downarrow 0$, $u^\epsilon \to u$ in $L^p_{\text{loc}}(I;X)$.

(v) Let $u \in L^p(I;X)$ be weakly differentiable with weak derivative $u' \in L^p(I;X)$. Show that
$$\frac{d}{dt}u_\epsilon(t) = \eta^\epsilon * u'(t), \quad \text{for a.e. } t \in (\epsilon, T-\epsilon),$$
$$\frac{d}{dt}u_\epsilon \to u' \quad \text{in } L^p_{\text{loc}}(I;X) \text{ as } \epsilon \downarrow 0.$$

Problem 4 (Sobolev spaces, 1+2+1 points).
For every $k = 1, 2, \dots$ and every $1 \le p \le \infty$, define the Sobolev space $W^{k,p}(I;X)$ as the set of equivalence classes of strongly measurable functions $u : I \to X$ with weak derivative $D_t^j u \in L^p(I;X)$ for $0 \le j \le k$. Define on $W^{k,p}(I;X)$

$$\|u\|_{W^{k,p}(I;X)} = \begin{cases} \left(\sum_{j=0}^k \|D_t^j u\|^p_{L^p(I;X)}\right)^{1/p}, & \text{for } p < \infty\,; \\ \sum_{j=0}^k \|D_t^j u(t)\|_{L^\infty(I;X)}, & \text{for } p = \infty\,. \end{cases}$$

(i) Show that $\|\cdot\|_{W^{k,p}(I;X)}$ is a norm.

(ii) Show that $W^{k,p}(I;X)$ is complete with respect to the norm $\|\cdot\|_{W^{k,p}(I;X)}$.

(iii) In the particular case when $X = \mathcal{H}$ is a Hilbert space and $p = 2$ show that the norm $\|\cdot\|_{W^{k,2}(I,\mathcal{H})}$ is induced by the scalar product

$$(u,v)_{H^k(I;\mathcal{H})} := \int_0^T \sum_{j=1}^k (D^j u(t), D^j v(t))_{\mathcal{H}}\, dt\,,$$

hence making $H^k(I,\mathcal{H}) := W^{2,k}(I;X)$ into a Hilbert space.

Problem 5 (Picard-Lindelöf theorem in Banach space, 1+2+2 points).
Let X be a Banach space and U an open set in X. Let $F : U \to X$ be a continuous function satisfying a **Lipschitz condition** on $U \subset X$, i.e there exists a number $K > 0$ such that

$$\|F(x) - F(y)\|_X \le K\|x - y\|_X\,, \qquad \text{for all } x, y \in U\,.$$

Let $x_0 \in U$ and $0 < a < 1$ such that $\overline{B}_{2a}(x_0) \subset U$, and $\sup_{x \in \overline{B}_{2a}(x_0)} \|F(x)\|_X < L < \infty$. Let $0 < b < \min\{a/L, 1/K\}$.

(i) Let $x \in B_a(x_0)$. Show that the set

$$M_x = \{u \in C([-b,b]; \overline{B}_{2a}(x_0)) \mid u(0) = x\}$$

is complete with respect to the sup norm.

(ii) Given a function $u \in M_x$ we define $S_x u : [-b,b] \to X$ by

$$S_x u(t) := x + \int_0^t F(u(s))ds.$$

Show that $s \mapsto F(u(s))$ is Bochner integrable and $S_x(M_x) \subset M_x$.

(iii) Show that for all $x \in B_a(x_0)$ there exists a unique function $u \in C([-b,b]; \overline{B}_{2a}(x_0))$ such that (1.) $u(0) = x$, (2.) u is differentiable with $\frac{d}{dt}u : (-b,b) \to X$, (3.) u is a solution of

$$\frac{d}{dt}u(t) = F(u(t)), \quad t \in (-b,b).$$

Hint: Use the Banach fixed point theorem.

Total: 20 points