Nonlinear Partial Differential Equations II | University of Bonn
Summer term 2017 | Prof. Dr. M. Disertori
Problem Sheet 11 (optional, due Wednesday 19.07.2017) | L. Borasi, M. Lager

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Problem 1 (Finite propagation speed for Hopf-Lax formula).
Let $F : \mathbb{R}^d \to \mathbb{R}$ be a convex function of class C^1, $g \in C^1$. Assume $\lim_{|v|\to\infty} \frac{F(v)}{|v|} = \infty$ and $F^* \in C^1$. Prove that the Hopf-Lax formula reads

$$u(x,t) = \min_{y\in\mathbb{R}^d}\left\{tF^*\left(\frac{x-y}{t}\right) + g(y)\right\} = \min_{y\in B(x,Rt)}\left\{tF^*\left(\frac{x-y}{t}\right) + g(y)\right\}$$

for $R = \sup_{y\in\mathbb{R}^d} |DF(Dg(y))|$. Here $B(x,Rt) = \{y \in \mathbb{R}^d : |y-x| < Rt\}$.

Problem 2 (Hopf-Lax formula - an example).
Use Hopf-Lax formula to solve explicitly the PDE

$$\begin{cases} \partial_t u + \frac{3}{4}(\partial_x u)^{\frac{4}{3}} = 0 & \text{for } (x,t) \in \mathbb{R}\times(0,\infty), \\ u(x,0) = x. \end{cases}$$

Problem 3 (Hopf-Lax formula for unbounded initial condition).
Let E be a closed subset of $\mathbb{R}^d$. Consider the initial-value problem

$$\begin{cases} \partial_t u + |Du|^2 = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^d \times \{t = 0\}, \end{cases} \tag{1}$$

where

$$g(x) = \begin{cases} 0 & \text{if } x \in E, \\ \infty & \text{if } x \notin E, \end{cases}.$$

(i) Assume we can apply the Hopf-Lax formula. Show that the solution is

$$u(x,t) = \frac{1}{4t}\operatorname{dist}(x,E)^2. \tag{2}$$

(ii) Is (2) a reasonable solution to (1)? Notice that, if ∂E is sufficiently smooth, there is a neighborhood U of E such that $\operatorname{dist}(x,E) := \inf_{y\in E}|x-y|$ is differentiable almost everywhere in $U\backslash E$ with $|D\operatorname{dist}(x,E)| = 1$.

Problem 4 (L^∞-contraction property of Hamilton-Jacobi equations).
Let $F \in C^2(\mathbb{R}^d)$ be uniformly convex with $\lim_{|v|\to\infty} \frac{F(v)}{|v|} = \infty$, and let $g_1, g_2 : \mathbb{R}^d \to \mathbb{R}$ be Lipschitz continuous. Assume u_1, u_2 are weak solutions of the initial-value problem

$$\begin{cases} \partial_t u_i + F(Du_i) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u_i = g_i & \text{on } \mathbb{R}^d \times \{t=0\}, \end{cases}$$

$i = 1, 2$. Prove the inequality

$$\|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le \|g_1 - g_2\|_{L^\infty(\mathbb{R}^d)} \qquad \text{for all } t > 0.$$

Problem 5 (Weak vs. strong solution).
Consider the PDE

$$\partial_t u + u\,\partial_x u = 0.$$

(i) Show that any C^1 solution of this equation is a C^1 solution of both equations

$$(a) \quad \partial_t u + \partial_x\left(\frac{u^2}{2}\right) = 0, \qquad (b) \quad \partial_t\left(\frac{u^2}{2}\right) + \partial_x\left(\frac{u^3}{3}\right) = 0.$$

(ii) Let

$$g(x) = \begin{cases} u_- & x < 0 \\ u_+ & x \ge 0 \end{cases},$$

for $u_-, u_+ \in \mathbb{R}$ fixed and such that $u_+ < u_-$.

By the method of characteristics find an entropy solution for (a) and for (b) with initial condition g.

Hint: Note that (b) is a conservation law for $w = u^2$.
