Nonlinear Partial Differential Equations II University of Bonn
Summer term 2017 Prof. Dr. M. Disertori
Problem Sheet 10 (due Wednesday 12.07.2017) L. Borasi, M. Lager

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Problem 1 (Legendre transform, 2+2+2 points).
For a function $F : \mathbb{R}^d \to \mathbb{R}$, denote by F^* its Legendre transform.

(i) Let $F(v) := \frac{1}{r}|v|^r$, for $1 < r < \infty$. Show that $F^*(p) = \frac{1}{r'}|p|^{r'}$, where $\frac{1}{r} + \frac{1}{r'} = 1$.

(ii) Let $F(v) := \frac{1}{2} v \cdot Av + b \cdot v$, where $A \in \mathbb{R}^{d\times d}$ is a symmetric, positive definite matrix, and $b \in \mathbb{R}^d$. Compute F^*.

(iii) Let $F(v) = \sqrt{1 + v^2}$, $v \in \mathbb{R}$. Compute F^*.

Problem 2 (A nonlinear example, 3 points).
Use the method of characteristics to solve the nonlinear initial value problem

$$\begin{cases} \partial_t u + (\partial_x u)^4 = 0 & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = \frac{3}{4} x^{\frac{4}{3}}. \end{cases}$$

Problem 3 (Subdifferential, 3 points).
Let $F : \mathbb{R}^d \to \mathbb{R}$ be a convex function with $F(v)/|v| \to \infty$ as $|v| \to \infty$. We say that $p \in \mathbb{R}^d$ belongs to the subdifferential of F at v, written $p \in \partial F(v)$, if

$$F(r) \geq F(v) + p \cdot (r - v) \qquad \text{for all } r \in \mathbb{R}^d.$$

Prove that

$$p \in \partial F(v) \quad \Longleftrightarrow \quad v \in \partial F^*(p) \quad \Longleftrightarrow \quad v \cdot p = F(v) + F^*(p),$$

where F^* is the Legendre transform of F.

Problem 4 (Conservation, 4 points).
Let $F \in C^\infty(\mathbb{R})$, $F(0) = 0$, and $g \in C^0_{\mathrm{c}}(\mathbb{R})$. Assume that u is a continuous integral solution to the conservation law

$$\begin{cases} \partial_t u + \partial_x F(u) = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

and u has compact support in $\mathbb{R} \times [0,\infty]$. Prove that

$$\int_{-\infty}^{+\infty} u(x,t)\,\mathrm{d}x = \int_{-\infty}^{+\infty} g(x)\,\mathrm{d}x \qquad \text{for all } t > 0.$$

Problem 5 (2+2 points).
Consider

$$\begin{cases} \partial_t u + \partial_x(e^u) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u = g & \text{on } \mathbb{R} \times \{t = 0\}, \end{cases}$$

with

$$g(x) = \begin{cases} 2 & \text{for } x < 0, \\ 1 & \text{for } x \geq 0. \end{cases}$$

(i) Construct an entropy solution using characteristics.

(ii) Sketch the characteristics in the (t, x)-plane.

Total: 20 points