Nonlinear Partial Differential Equations II University of Bonn
Summer term 2017 Prof. Dr. M. Disertori
Problem Sheet 1 (due Wednesday 03.05.2017) L. Borasi, M. Lager

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Problem 1 (Examples of Banach space-valued functions, 2+1 points).
Define $u_1 : (0,1) \to L^2(0,1)$ by $u_1(t) = \chi_{(0,t)}$ and $u_2 : (0,1) \to L^\infty(0,1)$ by $u_2(t) = \chi_{(0,t)}$.

(i) Prove if the functions u_1 and u_2 are strongly measurable.

(ii) Prove whether or not they are Bochner integrable and compute the integral if possible.

Let in the following be X a real Banach space and $I = (0,T)$ with $T > 0$ a bounded interval.

Problem 2 (Bochner's Theorem, 4+1 points).
Let $u : I \to X$ be a strongly measurable function.

(i) Prove Theorem 1.20, i.e., u is integrable if and only if $\|u(t)\|_X$ is integrable.

Hint: For the "if" part construct a suitable sequence of simple functions approximating u and apply dominated convergence.

(ii) Assume u is integrable. Prove that

$$\left\| \int_I u(t)dt \right\|_X \leq \int_I \|u(t)\|_X dt$$

Problem 3 (Properties of Bochner integrals, 2+2 points).
Let X, Y be real Banach spaces and $u : I \to X$ strongly measurable.

(i) Let $A : X \to Y$ be a bounded linear map and u integrable. Show that $Au : I \to Y$ is strongly measurable, integrable and

$$A\left[\int_I u(t)dt\right] = \int_I A[u(t)]dt.$$

(ii) Let $u \in L^1(I;X)$ and $\dim Y \geq 1$. Prove that the following two statements are equivalent:

(a) u is weakly differentiable with weak derivative $u' \in L^1(I;X)$.

(b) There exists $v \in L^1(I;X)$ such that for all $A : X \to Y$ bounded linear operators $Au \in L^1(I;Y)$ is weakly differentiable with $(Au)' = Av$.

Prove that if (b) holds then $u' = v$ a.e. in I.

Problem 4 (Dominated convergence, 3 points).
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence of functions, $u_n : I \to X$, such that

- u_n is strongly measurable and integrable for all $n \in \mathbb{N}$,
- $u_n(t)$ converge to $u(t)$ in X for a.e. $t \in I$,
- it exists a $\phi : I \to \mathbb{R}_+$ summable such that $\|u_n(t)\|_X \leq \phi(t)$ for all $n \in \mathbb{N}$ and a.e. $t \in I$.

Show that u is strongly measurable, integrable and

$$\lim_{n\to\infty} \int_I \|u_n(t) - u(t)\|_X dt = 0,$$
$$\lim_{n\to\infty} \int_I u_n(t)dt = \int_I u(t)dt.$$

Problem 5 (Lemmas 1.27 and 1.28, 3+2 points).
Let $u \in L^1_{\text{loc}}(I; X)$.

(i) Let $\int_I \xi(t)u(t)dt = 0$ for all $\xi \in C_c^\infty(I)$. Prove that $u = 0$ a.e. in I.

Hint: Use a sequence ξ_n of test functions s.t. ξ_n converges pointwise to $\chi_{[t,t+h]}$ and apply dominated convergence and Theorem 1.26.

(ii) Let $u \in L^1(I; X)$ be weakly differentiable and $u' = 0$. Show that $u(t)$ is constant a.e. in I.

Hint: Prove first that $\int_I w(t)u(t) = 0$ for $w \in C^\infty(\bar{I})$ with $\int_I w(t)dt = 0$ and use (i).

Total: 20 points