Mathematical aspects of phase transitions
Lecture Notes.

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Chapter 2

High temperature region

2.1 The O(N) model

Let us go back to the $O(N)$ model introduced in Chapter 1. The finite volume configuration set is denoted by

$$\Omega_\Lambda = (\mathcal{S}_N)^\Lambda = \{\mathbf{S}_\Lambda : \Lambda \to \mathcal{S}_N\}, \qquad \mathbf{S}_\Lambda(x) = S_x \in \mathcal{S}_N = \{S \in \mathbb{R}^N \,|\, \|S\| = 1\},$$

where $\Lambda = [-L, \dots, L]^d$, the energy functional is

$$H_\Lambda(\mathbf{S}_\Lambda) = -\tfrac{1}{2} \sum_{x,y\in\Lambda} J_{xy}(S_x, S_y) - \frac{1}{\beta} \sum_{x\in\Lambda} (h_x, S_x),$$

the interaction satisfies $J_{xy} = J_{yx} \geq 0$ and $h_x \in \mathbb{R}^N$ plays the role of a local magnetic field. Since $(S_x, S_x) = 1$ we can always fix $J_{xx} = 0$ up to a global multiplication factor. Without loss of generality we then set $J_{xx} = 0$ in the rest of this chapter.

In the following we denote by $\mathbf{h}_\Lambda = \{h_x\}_{x\in\Lambda}$ $\mathbf{h} = \{h_x\}_{x\in\mathbb{Z}^d}$ the magnetic field configurations in finite and infinite volume. Finally $h\mathbf{1}_\Lambda$, with $h \in \mathbb{R}^N$, will denote the constant magnetic field: $h_x = h \; \forall x \in \Lambda$. With these conventions the partition function is

$$Z_{\Lambda,\beta}(\mathbf{h}_\Lambda) = \int d\mathbf{S}_\Lambda \; e^{\frac{\beta}{2} \sum_{x,y\in\Lambda} J_{xy}(S_x,S_y)} e^{\sum_{x\in\Lambda}(h_x,S_x)}. \tag{2.1.1}$$

Boundary conditions: all the results/techniques we will see in this chapter work for any choice of the boundary conditions. In many cases we will consider periodic boundary conditions, i.e. the cube Λ is replaced by the torus $\Lambda_p = \mathbb{Z}^d/[-L, \dots, L]^d$. This choice has the advantage of respecting translation invariance.

Translation invariance: To simplify the formulas we will often consider translation invariant interactions i.e.

$$J_{x+z,y+z} = J_{x,y} \qquad \forall x, y, z \in \mathbb{Z}^d.$$

When periodic boundary conditions and a constant magnetic field are chosen the measure inside (2.1.1) is also translation invariant.

2.1.1 Observables

We are interested in the thermodynamic limit of the following functions.

Magnetization and susceptibility:

$$M^{\alpha}_{\Lambda,\beta}(h) = \tfrac{1}{|\Lambda|} \sum_{x\in\Lambda} \mathbb{E}^{\beta,h\mathbf{1}_\Lambda}_{\Lambda,N}[S_{x,\alpha}] = -\frac{\partial}{\partial_{h_\alpha}} \Phi_\Lambda(h\mathbf{1}_\Lambda) \qquad \alpha,\alpha' \in [1,\dots,N]$$

$$\chi^{\alpha,\alpha'}_{\Lambda,\beta}(h) = \tfrac{1}{|\Lambda|} \sum_{xy\in\Lambda} \mathbb{E}^{\beta,h\mathbf{1}_\Lambda}_{\Lambda,N}[S_{x,\alpha} S_{y,\alpha'}]_C = -\frac{\partial^2}{\partial_{h_\alpha}\partial_{h_{\alpha'}}} \Phi_\Lambda(h\mathbf{1}_\Lambda)$$

where the finite volume free anergy $\Phi_\Lambda(\mathbf{h}_\Lambda)$ is defined by

$$\Phi_{\Lambda,\beta}(\mathbf{h}_\Lambda) = -\frac{1}{|\Lambda|} \ln Z_{\Lambda,\beta}(\mathbf{h}_\Lambda)$$

Connected correlation functions:

$$\begin{aligned} \mathbb{E}^{\beta,h\mathbf{1}_\Lambda}_{\Lambda,N}\left[\prod_{j=1}^m S_{x_j,\alpha_j}\right]_C &= \left[\prod_{j=1}^m \frac{\partial}{\partial_{h_{x_j,\alpha_j}}}\right] \ln Z_{\Lambda,\beta}(\mathbf{h}_\Lambda)_{|\mathbf{h}_\Lambda = h\mathbf{1}_\Lambda} \\ &= \left[\prod_{j=1}^m \frac{\partial}{\partial_{h_{x_j,\alpha_j}}}\right] \Big[\ln Z_{\Lambda,\beta}(\mathbf{h}_\Lambda) - \ln Z_{\Lambda,\beta}(h\mathbf{1}_\Lambda)\Big]_{|\mathbf{h}_\Lambda = h\mathbf{1}_\Lambda} \end{aligned}$$

where $\alpha_j \in [1,\dots,N]$.

The behavior of these limits depends on the regularity properties of the logarithm of the partition function as $\Lambda \to \mathbb{Z}^d$. We remark that the functions above are obtained by

1. either fixing a constant magnetic field on Λ $h_x = h$ $\forall x \in \Lambda\mathbb{Z}^d$ and taking derivatives with respect to h_α of the *free energy* $\Phi_{\Lambda,\beta}(h\mathbf{1}_\Lambda)$ (magnetization and susceptibility); note that $h\mathbf{1}_\Lambda$ can be directly extended to an infinite volume magnetic field configuration $h\mathbf{1}_{\mathbb{Z}^d}$;

2. or taking *local* variations of a *constant* magnetic field i.e. there exists some *finite* set X_h and a vector $h \in \mathbb{R}^N$ such that $h_x = h$ $\forall x \in \Lambda\backslash X$ (since X_h

is finite this is always true if Λ is large enough), and taking derivatives in h_{x,α_x}, $x \in X_h$ of the function

$$\ln \frac{Z_{\Lambda,\beta}(\mathbf{h}_\Lambda)}{Z_{\Lambda,\beta}(h\mathbf{1}_\Lambda)}$$

(connected correlation functions); in this last case there is no $1/|\Lambda|$ factor, the partition function is replaced by a ratio and $X_h = \{x_1, \ldots, x_m\}$.

To encode these cases we extend the finite volume partition function $Z_{\Lambda,\beta}(\mathbf{h}_\Lambda)$ to a function on *infinite* volume magnetic field configurations belonging to the set

$$H = \{\mathbf{h} \in (\mathbb{R}^N)^{\mathbb{Z}^d} \,|\, \exists X_{\mathbf{h}} \subset \mathbb{Z}^d, |X_{\mathbf{h}}| < \infty \text{ and } \exists h \in \mathbb{R}^N, \text{ with } h_x = h \,\forall x \in \mathbb{Z}^d \backslash X_{\mathbf{h}}\}, \tag{2.1.2}$$

by

$$\begin{aligned} Z_{\Lambda,\beta}: \quad & H \to \mathbb{R} \\ & \mathbf{h} \to Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda}) \end{aligned}$$

where $\mathbf{h}_{|\Lambda} \in (\mathbb{R}^N)^\Lambda$ is defined by $\mathbf{h}_{|\Lambda} = \{\mathbf{h}(x)\}_{x\in\Lambda}$.

2.1.2 Main result

We will prove the following result.

Theorem 1 *For any $N \geq 1$ there exist constants $0 < \beta_0 < 1$, $0 < \varepsilon < 1$ such that $\forall \beta \leq \beta_0$*

$$\Phi_\beta(h\mathbf{1}_{\mathbb{Z}^d}) = \lim_{\Lambda\to\mathbb{Z}^d} \Phi_{\Lambda,\beta}(h\mathbf{1}_\Lambda)$$

exists and is analytic for $h \in B_{\mathbb{C}^N}(0,\varepsilon)$, where $B_{\mathbb{C}^N}(0,\varepsilon) = \{v \in \mathbb{C}^N |\; \|v\| < \varepsilon\}$. Moreover

$$\left[\prod_{\alpha=1}^N (\partial_{h_\alpha})^{n_\alpha}\right] \Phi_\beta(h\mathbf{1}_{\mathbb{Z}^d}) = \lim_{\Lambda\to\mathbb{Z}^d} \left[\prod_{\alpha=1}^N (\partial_{h_\alpha})^{n_\alpha}\right] \Phi_{\Lambda,\beta}(h\mathbf{1}_\Lambda)$$

for any choice of $n_i \in \mathbb{N}$. The same statements hold for

$$F_\beta(\mathbf{h}) = \lim_{\Lambda\to\mathbb{Z}^d} \ln \frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,\beta}(h\mathbf{1}_\Lambda)}.$$

for any $\mathbf{h} \in H$ with $\mathbf{h}(x) = h \;\forall x \notin Y$, where Y is any fixed finite set.

Consequence. This result implies in particular that the free energy and all its derivatives are continuous functions of h near $h = 0$, therefore there can be no phase transition in this temperature range.

Proof of Theorem 1 Let $\mathbf{h} \in H$ with $\mathbf{h}(x) = h \ \forall x \notin X_{\mathbf{h}}$. By high temperature expansions (see Section 2.4) $\ln Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})$ can be written as a sum of local functions

$$\ln Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda}) = \sum_{X \subset \Lambda} K_{X,\beta}(\mathbf{h}_{|X})$$

where $K_{X,\beta}$ is an infinite sum of functions F_n, where each F_n is (complex) analytic with respect to $\{h_x \in \mathbb{C}^N\}_{x \in X}$ as long as $\|h_x\| \leq \varepsilon \ \forall x \in X$:

$$K_{X,\beta}(\mathbf{h}_{|X}) = \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n \subset X \\ \cup_{i=1}^n X_i = X}} F_{n,\beta}(X_1, \ldots, X_n; \mathbf{h}_{|X}).$$

In Section 2.5 we will see that, for $\beta \leq \beta_0$, this sum is absolutely convergent *uniformly* in $h \in H \cap B_{(\mathbb{C}^N)^{\mathbb{Z}^d}}(0, \varepsilon)$. Then by Vitali's theorem[1] the limit function $K_{X,\beta}$ is analytic in $\{h_x\}_{x \in X}$. Now if J_{xy} is translation invariant K satisfies $K_{X,\beta}(h\mathbf{1}_X) = K_{T_z(X),\beta}(h\mathbf{1}_{T_z(X)}) \ \forall z \in \mathbb{Z}^d$, where

$$\begin{aligned} T: \quad & \mathbb{Z}^d \to \mathbb{Z}^d \\ & x \to T_z(x) = x + z \end{aligned} \qquad T_z(X) = \{T_z(x)\}_{x \in X}.$$

This will become clear from the expression we will obtain for K. Then

$$\begin{aligned} \lim_{\Lambda \to \mathbb{Z}^d} \Phi_{\Lambda,\beta}(h\mathbf{1}_\Lambda) &= \lim_{\Lambda \to \mathbb{Z}^d} \tfrac{1}{|\Lambda|} \sum_{X \subset \Lambda} K_{X,\beta}(h\mathbf{1}_X) = \lim_{\Lambda \to \mathbb{Z}^d} \tfrac{1}{|\Lambda|} \sum_{X \subset \Lambda} \sum_{x_0 \in X} \tfrac{K_{X,\beta}(h\mathbf{1}_X)}{|X|} \\ &= \lim_{\Lambda \to \mathbb{Z}^d} \tfrac{1}{|\Lambda|} \sum_{x_0 \in \Lambda} \sum_{\substack{X \subset \Lambda \\ x_0 \in X}} \tfrac{K_{X,\beta}(h\mathbf{1}_X)}{|X|} = \sum_{\substack{X \subset \mathbb{Z}^d, |X| < \infty \\ 0 \in X}} \tfrac{K_{X,\beta}(h\mathbf{1}_X)}{|X|} \end{aligned} \tag{2.1.3}$$

where in the last line we used translation invariance:[2].

$$K_{X,\beta}(h\mathbf{1}_X) = K_{T_{-x_0}(X),\beta}(h\mathbf{1}_{T_{-x_0}(X)}).$$

Moreover

$$\begin{aligned} \lim_{\Lambda \to \mathbb{Z}^d} [\ln Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda}) - \ln Z_{\Lambda,\beta}(h\mathbf{1}_{|\Lambda})] &= \lim_{\Lambda \to \mathbb{Z}^d} \sum_{\substack{X \subset \Lambda \\ X \cap X_{\mathbf{h}} \neq \emptyset}} [K_{X,\beta}(\mathbf{h}_{|X}) - K_{X,\beta}(h\mathbf{1}_X)] \\ = \sum_{\substack{X \subset \mathbb{Z}^d, |X| < \infty \\ X \cap X_{\mathbf{h}} \neq \emptyset}} [K_{X,\beta}(\mathbf{h}_{|X}) - K_{X,\beta}(h\mathbf{1}_X)] \end{aligned} \tag{2.1.4}$$

[1] Convergence Vitali Theorem: Let D be an open connected set of $\mathbb{C}$, f_n a sequence of analytic functions in D, locally uniformly bounded such that the sequence converges on a set with some accumulation point in D. Then f_n converges locally uniformly in D. The limit is then an analytic function.

[2] In the non translation invariant case this equality becomes an inequality:

$$|\lim_{\Lambda \to \mathbb{Z}^d} \Phi_{\Lambda,\beta}(h\mathbf{1}_\Lambda)| \leq \sup_{x_0 \in \mathbb{Z}^d} \left| \sum_{\substack{X \subset \mathbb{Z}^d, |X| < \infty \\ x_0 \in X}} \frac{K_{X,\beta}(h\mathbf{1}_X)}{|X|} \right|$$

since $K_{X,\beta}(\mathbf{h}_{|X}) = K_{X,\beta}(h\mathbf{1}_X)\ \forall X \cap X_{\mathbf{h}} = \emptyset$, where $X_{\mathbf{h}}$ is a *finite* set depending on $\mathbf{h}$. This result holds also for non translation invariant interactions. We will see in the next section that (2.1.3) converges uniformly in $h \in B_{\mathbb{C}^N}(0,\varepsilon)$ and (2.1.4) converges uniformly in

$$\{\mathbf{h} \in H |\, \mathbf{h} \in B_{(\mathbb{C}^N)^{\mathbb{Z}^d}}(0,\varepsilon) \text{ and } X_{\mathbf{h}} = Y\}$$

for any fixed *finite* set Y. Then both limits are analytic functions. Finally, by Cauchy formula ($\frac{2\pi i}{n!} f^n(x) = \oint_\gamma \frac{f(z)}{(z-x)^{n+1}}$) and analyticity of the limit functions, the thermodynamic limit ($\lim_{\Lambda\to\mathbb{Z}^d}$) commutes with all derivatives in h. □

2.2 The limit case: infinite temperature.

When $\beta = 0$ (i.e. $T = \infty$) the partition function factors

$$Z_{\Lambda,0}(\mathbf{h}_\Lambda) = \prod_{x\in\Lambda} \int dS_x e^{(h_x,S_x)} = \prod_{x\in\Lambda} Z_{x,0}(h_x) = \prod_{x\in\Lambda} Z_{0,0}(h_x),$$

where $Z_{x,0}(h_x) = Z_{0,0}(h_x)$ ($x = 0$ is the origin in $\mathbb{Z}^d$) is analytic on $\mathbb{C}^N$ and *bounded away from zero* for $h_x \in B_{\mathbb{C}^N}(0,\varepsilon)$. Therefore $\ln Z_{x,0}(h_x)$ is analytic in $B_{\mathbb{C}^N}(0,\varepsilon)$. In the same way for each $Y \subset \mathbb{Z}^d$ with $|Y| < \infty$ the function

$$\ln Z_{\Lambda,0}(\mathbf{h}_{|\Lambda}) = \sum_{x\in Y} \ln Z_{x,0}(h_x) + (|\Lambda| - |Y|) \ln Z_{0,0}(h)$$

is analytic in $\{\mathbf{h} \in H | X_{\mathbf{h}} = Y \text{ and } \mathbf{h} \in B_{(\mathbb{C}^N)^{\mathbb{Z}^d}}(0,\varepsilon)\}$ and

$$\lim_{\Lambda\to\mathbb{Z}^d} \Phi_{\Lambda,0}(h\mathbf{1}_\Lambda) = \ln Z_{0,0}(h_x) = \ln \int dS_0 e^{(h,S_0)}$$
$$\lim_{\Lambda\to\mathbb{Z}^d} [\ln Z_{\Lambda,0}(\mathbf{h}_{|\Lambda}) - \ln Z_{\Lambda,0}(h\mathbf{1}_{|\Lambda})] = \sum_{x\in Y} \ln \tfrac{Z_{x,0}(h_x)}{Z_{x,0}(h)}.$$

are analytic functions too. When $0 < \beta < 1$ we expand around the case $\beta = 0$. Therefore in the following we concentrate on

$$\frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} = \frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{\prod_{x\in\Lambda} Z_{x,\beta}(h_x)} \tag{2.2.5}$$

where we used $Z_{x,\beta}(h_x) = Z_{x,0}(h_x)$.

2.3 Preliminary combinatorial definitions and results.

For simplicity we state all definitions and results using Λ and X finite subsets of $\mathbb{Z}^d$, but everything can be generalized to any (finite or infinite) abstract set of points (not necessarily in $\mathbb{Z}^d$).

Edges and paths. The set $\mathcal{E}_\Lambda$ of all possible *undirected edges* connecting pairs of points in Λ is denoted by

$$\mathcal{E}_\Lambda = \{(x,y)|x \neq y \in \Lambda\}, \qquad \text{with } (x,y) = (y,x).$$

A *simple path* γ_{ab} of lenght n starting at a and ending at b is a subset of $\mathcal{E}_\Lambda$

$$\gamma_{ab} = \{e_1, \ldots, e_n\} \subset \mathcal{E}_\Lambda$$

such that there exist $x_1, \ldots x_{n-1}$ points in Λ satisfying

$$e_1 = (a, x_1),\ e_2 = (x_1, x_2), \ldots, e_n = (x_{n-1}, b).$$

With this definition each edge of $\mathcal{E}_\Lambda$ can appear only once in the path (we never travel on the same edge twice). When $a = b$ we say that γ_{aa} is a closed path (also called *cycle* or *loop*).

Graphs. A *graph* G is a pair (X, E), where $X \subset \Lambda$ and $E \subset \mathcal{E}_X$. A graph $G = (X, E)$ is *connected* if for any pair $a, b \in X$ there is a simple path γ_{ab} made of edges in E connecting a to b. A graph $G = (X, E)$ with no cycle is called a *forest on* X. A connected graph $G = (X, E)$ with no cycle is called a *tree on* X. We will denote by

$$\begin{aligned} \mathcal{G}[X] &= \{G = (X,E)|\ E \subset \mathcal{E}_X\} \qquad & \mathcal{F}[X] &= \{G \in \mathcal{G}[X]|\ G \text{ has no cycle}\} \\ \mathcal{G}_c[X] &= \{G \in \mathcal{G}[X]|\ G \text{ connected}\} \qquad & \mathcal{T}[X] &= \{G \in \mathcal{G}_c[X]|\ G \text{ has no cycle}\} \end{aligned}$$

the set of graphs, connected graphs, forests and trees on a fixed set X.

Connected components. Each graph $G = (X, E)$ can be associated to a unique partition $P \in \mathcal{P}[X]$ of X into subsets such that

$$\begin{aligned} &(Y, E \cap \mathcal{E}_Y) \in \mathcal{G}_c[Y],\ \forall Y \in P, \quad \text{and} \\ &\qquad\qquad a \in Y, b \in Y',\ \text{with } Y, Y' \in P, Y \neq Y' \Rightarrow (a,b) \notin E. \end{aligned}$$

The elements Y in the partition are called *connected components of* G.

Characterization of a graph. A graph $G = (X, E)$ can be uniquely determined by giving the following information:

1. a partition $P \in \mathcal{P}[X]$ (fixes the connected components) and
2. for each $Y \in P$ a connected graph $g \in \mathcal{G}_c[Y]$ (this fixes the edges inside each connected component).

In the same way a forest $F \in \mathcal{F}[X]$ is uniquely determined by giving the following information:

1. a partition $P \in \mathcal{P}[X]$ (fixes the connected components) and
2. for each $Y \in P$ a tree $T \in \mathcal{T}[Y]$ (this fixes the edges inside each connected component).

Functions and connected functions. Let $\mathcal{P}a_\Lambda = \{X \subset \Lambda\}$ the set of all subsets (parts) of Λ. Let ψ be some function on $\mathcal{P}a_\Lambda$

$$\psi : \begin{array}{l} \mathcal{P}a_\Lambda \to \mathbb{R} \\ X \to \psi(X) \end{array} .$$

Lemma 1 *There exists a unique function $\psi_c : \mathcal{P}a_\Lambda \to \mathbb{R}$ satisfying the equation*

$$\psi(X) = \sum_{P \in \mathcal{P}[X]} \prod_{Y \in P} \psi_c(Y) \qquad \forall X \in \mathcal{P}a_\Lambda. \tag{2.3.6}$$

Proof. The proof is done by induction on the size of X. Let $|X| = 1$, then $\mathcal{P}[X] = X$ and $\psi_c(X) = \psi(X)$ is the unique solution. Now let us suppose there is a unique solution for the equation for any set Y with $|Y| = n$. Let $|X| = n+1$. Then (2.3.6) can be writte an

$$\psi(X) = \psi_c(X) + \sum_{P \in \mathcal{P}[X] \setminus X} \prod_{Y \in P} \psi_c(Y)$$

The second term contains only sets Y with $|Y| \leq n$ then by the induction hypothesis all factors $\psi_c(Y)$ have been uniquely determined already. It remains only $\psi_c(X)$, hence

$$\psi_c(X) = \psi(X) - \sum_{P \in \mathcal{P}[X] \setminus X} \prod_{Y \in P} \psi_c(Y)$$

is the unique possible choice. This ends the proof. $\square$

2.4 Setting up the expansion

Lemma 2 *The finite volume partition function can be written as*

$$Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda}) = \sum_{P \in \mathcal{P}[\Lambda]} \prod_{X \in P} A(X), \qquad \textit{with} \tag{2.4.7}$$

$$A(X) = \int \prod_{x \in X} d\nu_x(S_x) w_c(X), \tag{2.4.8}$$

where $\mathcal{P}[\Lambda]$ is the set of all possible partitions of Λ into (not necessarily connected) subsets,

$$d\nu_x(S_x) = dS_x e^{(h_x, S_x)},$$

the interaction term $w_c(X)$ is the unique solution of (2.3.6) *with $\psi(X)$ replaced by*

$$w(X) = \prod_{x<y \in X} e^{\beta J_{xy}(S_x, S_y)}$$

and we have introduced some fixed ordering $<$ on the set X.

Proof. With the definitions above the partition function is written as

$$Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda}) = \int \prod_{x\in\Lambda} d\nu_x(S_x) w(\Lambda).$$

By (2.3.6) there exists a unique function w_c solution of

$$w(X) = \sum_{P\in\mathcal{P}[X]} \prod_{Y\in P} w_c(Y).$$

Each function $w_c(Y)$ depends only on spins S_x with $x \in Y$. Therefore the integral factors

$$\begin{aligned} Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda}) &= \sum_{P\in\mathcal{P}[X]} \int \prod_{x\in X} d\nu_x(S_x) \prod_{Y\in P} w_c(Y) \\ &= \sum_{P\in\mathcal{P}[X]} \prod_{Y\in P} \left[\int \prod_{x\in Y} d\nu_x(S_x) w_c(Y) \right]. \end{aligned}$$

This ends the proof. □

Lemma 3 *The ratio of partition functions* (2.2.5) *can be written as*

$$\frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda \\ |X_i|\geq 2,\ \forall i}} \left[\prod_{i=1}^{n} \mathcal{A}(X_i) \right] V(X_1,\ldots,X_n) \tag{2.4.9}$$

where the amplitude is defined as

$$\mathcal{A}(X) = \frac{A(X)}{\prod_{x\in X} A(\{x\})}, \qquad A(\{x\}) = \int d\nu(S_x) = \int dS_x e^{(h_x,S_x)},$$

and V implements the non-overlapping condition for the sets X_i

$$V(X_1,\ldots,X_n) = \prod_{i<j} V(X_i,X_j), \qquad V(X,X') = \begin{cases} 1 & X\cap X' = \emptyset \\ 0 & X\cap X' \neq \emptyset \end{cases} \tag{2.4.10}$$

Proof. We remark that

$$\prod_{x\in\Lambda} Z_{x,0}(h_x) = \prod_{X\in P} \prod_{x\in X} Z_{x,0}(h_x), \qquad \text{and} \quad Z_{x,0}(h_x) = A(\{x\}).$$

Then using Lemma 2 and (2.2.5) we have

$$\frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} = \sum_{P\in\mathcal{P}[\Lambda]} \prod_{X\in P} \frac{A(X)}{\prod_{x\in X} A(\{x\})} = \sum_{P\in\mathcal{P}[\Lambda]} \prod_{X\in P} \mathcal{A}(X)$$

$$= \sum_{P\in\mathcal{P}[\Lambda]} \prod_{\substack{X\in P\\ |X|\geq 2}} \mathcal{A}(X) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda\\ |X_i|\geq 2, X_i\cap X_j=\emptyset\forall i>j}} \left[\prod_{i=1}^n \mathcal{A}(X_i)\right]$$

$$= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda\\ |X_i|\geq 2,\ \forall i}} \left[\prod_{i=1}^n \mathcal{A}(X_i)\right] V(X_1,\ldots,X_n)$$

where we used $\mathcal{A}(X) = 1$ whenever $|X| = 1$. This ends the proof. □

Functions and connected functions. For each given sequence of subsets $\mathcal{X} = \{X_i\}_{i\in\mathbb{N}}$, $X_i \subset \Lambda$, the interaction V defined in (2.4.10) can be seen as a function on $\mathcal{P}a_{\mathbb{N}} = \{I \subset \mathbb{N}\}$:

$$\begin{array}{rl} V: & \mathcal{P}a_{\mathbb{N}} \to \mathbb{R} \\ & I \to V(I) = V(\{X_i\}_{i\in I}) \end{array} .$$

Then by Lemma 2 there is a unique function $V_c : \mathcal{P}a_{\mathbb{N}} \to \mathbb{R}$ such that

$$V(I) = \sum_{\mathcal{I}\in\mathcal{P}[I]} \prod_{I'\in\mathcal{I}} V_c(I'). \tag{2.4.11}$$

With this definition we can state the main result.

Lemma 4 *The logarithm of the ratio* (2.2.5) *can be* **formally** *written as*

$$\ln \frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} = \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda\\ |X_i|\geq 2,\ \forall i}} \left[\prod_{i=1}^n \mathcal{A}(X_i)\right] V_c(X_1,\ldots,X_n) \tag{2.4.12}$$

where, V_c *is defined in* (2.4.11) *above.*

Proof. By (2.4.9) and (2.4.11) above we have

$$\frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda\\ |X_i|\geq 2,\ \forall i}} \left[\prod_{i=1}^n \mathcal{A}(X_i)\right] \sum_{\mathcal{I}\in\mathcal{P}[\{1,\ldots n\}]]} \prod_{I\in\mathcal{I}} V_c(I)$$

$$= 1 + \sum_{n\geq 1} \sum_{\mathcal{I}\in\mathcal{P}[\{1,\ldots n\}]} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda\\ |X_i|\geq 2,\ \forall i}} \left[\prod_{i=1}^n \mathcal{A}(X_i)\right] \prod_{I\in\mathcal{I}} V_c(I)$$

$$= 1 + \sum_{n\geq 1} \sum_{\mathcal{I}\in\mathcal{P}[\{1,\ldots n\}]} \frac{1}{n!} \prod_{I\in\mathcal{I}} \mathbf{A}(I)$$

where

$$\mathbf{A}(I) = \sum_{\substack{X_i \subset \Lambda, i \in I \\ |X_i| \geq 2}} V_c(I) \prod_{i \in I} \mathcal{A}(X_i) = \sum_{\substack{X_i \subset \Lambda, i \in I \\ |X_i| \geq 2}} V_c(\{X_i\}_{i \in I}) \prod_{i \in I} \mathcal{A}(X_i)$$

$$= \sum_{\substack{X_1, \ldots, X_{|I|} \subset \Lambda \\ |X_i| \geq 2}} V_c(X_1, \ldots, X_{|I|}) \prod_{i=1}^{|I|} \mathcal{A}(X_i) = \mathbf{A}(|I|)$$

and in the last line we relabelled the sets $\{X_i\}_{i \in I}$. The expression then depends only on the number of elements inside I.

Each partition $\mathcal{I} \in \mathcal{P}[\{1, \ldots n\}]$ can be uniquely identified by the three following ingredients:

1. the number $k = |\mathcal{I}|$ of components in the partition $1 \leq k \leq n$;
2. the cardinals of each component i.e. k numbers $n_1, \ldots n_k$ satisfying $n_i \geq 1$ and $\sum_{j=1}^k n_j = n$;
3. k subsets $I_1, \ldots, I_k$ of $\{1, \ldots n\}$ satisfying $I_j = n_j$ $\forall j = 1, \ldots, k$ and $I_j \cap I_{j'} = \emptyset$ $\forall j \neq j'$.

Then $\mathbf{A}(I)$ depends only on $|I| = n_I$. The formula above becomes

$$\frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} = 1 + \sum_{n \geq 1} \sum_{k=1}^{n} \sum_{\substack{n_1, \ldots n_k \geq 1 \\ \sum_{j=1}^k n_j = n}} \sum_{\substack{I_1, \ldots I_k \subset \Lambda, |I_j| = n_j \\ I_j \cap I_{j'} = \emptyset \forall j \neq j'}} \frac{1}{n!} \prod_{j=1}^{k} \mathbf{A}(n_j)$$

$$= 1 + \sum_{n \geq 1} \sum_{k=1}^{n} \frac{1}{k!} \sum_{\substack{n_1, \ldots n_k \geq 1 \\ \sum_{j=1}^k n_j = n}} \prod_{j=1}^{k} \frac{1}{n_j!} \mathbf{A}(n_j) = 1 + \sum_{k \geq 1} \frac{1}{k!} \mathbf{A}^k = e^{\mathbf{A}}$$

where we defined

$$\mathbf{A} = \sum_{n \geq 1} \frac{1}{n!} \mathbf{A}(n)$$

and we used the following relations

$$\mathrm{card}\Big\{ I_1, \ldots I_k \subset \Lambda \mid |I_j| = n_j \forall j, I_j \cap I_{j'} = \emptyset \ \forall j \neq j' \Big\} = \frac{n!}{k! \prod_{j=1}^k n_j!}$$

$$\sum_{n \geq 1} \sum_{k=1}^{n} (\cdot) = \sum_{k \geq 1} \sum_{n \geq k} (\cdot), \qquad \sum_{n \geq k} \sum_{\substack{n_1, \ldots n_k \geq 1 \\ \sum_{j=1}^k n_j = n}} (\cdot) = \sum_{n_1, \ldots n_k \geq 1} (\cdot)$$

Therefore $\mathbf{A}$ is the formal logarithm of the ratio of two partition functions. Inserting the definition of $\mathbf{A}(n)$ we obtain

$$\mathbf{A} = \sum_{n \geq 1} \frac{1}{n!} \sum_{\substack{X_1, \ldots, X_n \subset \Lambda \\ |X_i| \geq 2}} V_c(X_1, \ldots, X_n) \prod_{i=1}^{n} \mathcal{A}(X_i).$$

This ends the proof. □

Remark 1. While the sums (2.4.7) and (2.4.9) are finite, the expression in (2.4.12) is an infinite sum, hence the formula is exact (and not just formal) if we can prove that the sum is absolutely convergent. This will be done in the next subsection.

Remark 2. The connected functions w_c and V_c appearing in (2.4.7) and (2.4.12) can be expressed as sums over connected graphs. This is showed in the next lemma.

Lemma 5 *The connected functions w_c and V_c in the lemmas above can be expressed as*

$$w_c(X) = \sum_{G \in \mathcal{G}_c[X]} \prod_{e \in E} [w_e - 1],$$

$$V_c(X_1, \ldots, X_n) = \sum_{G \in \mathcal{G}_c[\{1,\ldots,n\}]} \prod_{e \in E} [V_e - 1]$$

where

$$w_e = w(x_e, y_e) = e^{\beta J_{x_e y_e}(S_{x_e}, S_{y_e})}$$

$$V_e = V(i_e, j_e) = V(X_{i_e}, X_{j_e}) = \begin{cases} 1 & X_{i_e} \cap X_{j_e} = \emptyset \\ 0 & X_{i_e} \cap X_{j_e} \neq \emptyset \end{cases}$$

Proof The interaction term in the partition function can be written as

$$w(\Lambda) = e^{\frac{\beta}{2}\sum_{x,y \in \Lambda} J_{xy}(S_x, S_y)} = \prod_{e \in E_\Lambda} w_e = \prod_{e \in E_\Lambda} [1 + (w_e - 1)] = \sum_{G=(X,E) \in \mathcal{G}[\Lambda]} \prod_{e \in E} [w_e - 1]$$

where we used $J_{xx} = 0$. Now each graph can be uniquely determined fixing a partition $P \in \mathcal{P}[\Lambda]$ and a connected graph $g \in \mathcal{G}_c[X]$ for each component $X \in P$ (see subsect.2.3). Then

$$w(\Lambda) = \sum_{P \in \mathcal{P}[\Lambda]} \prod_{X \in P} \left[\sum_{g=(X,E) \in \mathcal{G}_c[X]} \prod_{e \in E} (w_e - 1) \right].$$

This is then a solution of $w(\Lambda) = \sum_{P \in \mathcal{P}[\Lambda]} \prod_{X \in P} w_c(X)$. The solution being unique we have the result. The same arguments apply for V. □

2.5 Convergence of the log expansion

To prove the convergence of (2.4.12) we will need three separate steps. The first (Thm. 2 below) depends on the details of the model we are considering, while the last two are model independent.

Theorem 2 *There exists a $\beta_0 > 0$ such that $\forall \beta < \beta_0$*

$$\sup_{a\in\mathbb{Z}^d} \sum_{X\subset\mathbb{Z}^d, 2\le|X|<\infty, a\in X} |\mathcal{A}(X)|e^{|X|} < 1. \tag{2.5.13}$$

The result is true for any $d \geq 1$, for all $N \geq 1$ and for any interaction J_{xy} such that $\sup_{x\in\mathbb{Z}^d} \sum_{y\in\mathbb{Z}^d} J_{xy} < \infty$.

Theorem 3 *If* (2.5.13) *holds then for any fixed point $a \in \Lambda$ the sum*

$$\sum_{n\geq1} \tfrac{1}{(n-1)!} \sum_{\substack{X_1\subset\Lambda \\ |X_1|\geq2, a\in X_1}} \frac{|\mathcal{A}(X_1)|}{|X_1|} \sum_{\substack{X_2,\ldots,X_n\subset\Lambda \\ |X_i|\geq2\forall i}} \left[\prod_{i=2}^n |\mathcal{A}(X_i)|\right] |V_c(X_1,\ldots,X_n)| \tag{2.5.14}$$

is convergent uniformly in the volume Λ and in the point a.

Theorem 4 *If* (2.5.14) *holds then the series*

$$\frac{1}{|\Lambda|} \ln \frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} = \frac{1}{|\Lambda|} \sum_{n\geq1} \frac{1}{n!} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda \\ |X_i|\geq2,\ \forall i}} \left[\prod_{i=1}^n \mathcal{A}(X_i)\right] V_c(X_1,\ldots,X_n)$$

converges in absolute value uniformly in Λ.

Remark. The same arguments hold to prove convergence of the analog expression for $\ln \frac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,\beta}(h\mathbf{1}_{|\Lambda})}$.

2.6 Cluster expansion: special case

In this section we will prove Theorem 2 in the special case when

$$\sup_x \sum_{xy} J_{xy}^{1/2} < \infty.$$

Let $h_x \in B_{\mathbb{C}^N}(0,\varepsilon)$. Then

$$|\mathcal{A}(X)| \leq \int \prod_{x\in X} \frac{dS_x e^{(\Re h_x, S_x)}}{|A(\{x\})|} |w_c(X)|$$

Since h_x is small we have

$$|A(\{x\})| = |\int dS_x e^{(h_x,S_x)}| = 1 + O(\varepsilon^2), \qquad e^{(\Re h_x, S_x)} = 1 + O(\varepsilon).$$

Moreover

$$|w_c(X)| \leq \sum_{g=(X,E)\in\mathcal{G}_c[X]} \prod_{e\in E} |w_e - 1| \leq \sum_{g=(X,E)\in\mathcal{G}_c[X]} \prod_{e\in E} \beta J_e e^{\beta J_e}$$

where we used $|e^x - 1| \le |x|e^{|x|}$, $|(S_x, S_y)| \le 1$ and we abbreviated $J_e = J_{x,y}$ for each edge $e = (x, y)$. Then

$$|\mathcal{A}(X)| \le K_\varepsilon^{|X|} \sum_{g=(X,E)\in\mathcal{G}_c[X]} \prod_{e\in E} \beta J_e e^{\beta J_e}.$$

where $K_\varepsilon = 1 + O(\varepsilon)$. We will need the following two remarks.

Remark 1. Choosing a connected graph $g = (X, E)$ is equivalent to

1. choose a set of points X and
2. choose a set $E \subset E_X$ such that (X, E) is connected.

Definition. We define a *path* in Λ of lenght n starting at a as a set of $n+1$ points $\gamma = \{x_0, \ldots, x_n\}$ such that $x_0 = a$. The path may use the same edge of E_Λ several times. Let $\mathcal{W}_a^n$ the set of paths starting at a of lenght n.

Remark 2. Each connected graph $g = (X, E)$ with $a \in X$ and $|X| \ge 2$ can be associated to a (non unique) path $\gamma_g = (x_0, .., x_{2|E|}) \in \mathcal{W}_a^{2|E|}$ starting at a of lenght $n = 2|E|$ such that

1. $(x_j, x_{j+1}) \in E$ $\forall j = 0, \ldots, n-1$ (the path contains only edges in E) and
2. each edge $e = (x, y) \in g$ appears exactly twice in the path (crossed in opposite directions): $\exists j \ne k$ such that $(x_j, x_{j+1}) = (x, y)$ and $(x_k, x_{k+1}) = (y, x)$.

This can be seen by induction on the size of the set (see [3]). When $X = \{x, y\}$ contains only two points then the only connected graph is $e = (x, y)$ and the unique path is given by $\gamma = \{e_1, e_2\}$ with $e_1 = (x, y)$ and $e_2 = (y, x)$. Now suppose the statement is true for $G = (X, E)$ a connected graph on X and let γ_G be the corresponding path. Let us add one point y to the set X and let E' the set of lines we add to E in order to obtain G' a connected set on $X \cup \{y\}$. To construct a path $\gamma_{G'}$ we choose one edge in E' and add it to γ_G twice, so that we cross it in both directions. This completes the construction.

Now with these remarks we can write

$$\begin{aligned}
\sum_{\substack{X\subset\mathbb{Z}^d, a\in X\\ 2\le|X|<\infty}} |\mathcal{A}(X)|e^{|X|} &\le \sum_{\substack{g=(X,E)\in\mathcal{G}_c[X]\\ |X|\ge 2, a\in X}} [eK_\varepsilon]^{|X|} \prod_{e\in E} \beta J_e e^{\beta J_e} \\
&= \sum_{n\ge 1} \sum_{\substack{g=(X,E)\in\mathcal{G}_c[X]\\ |E|=n,, a\in X}} [eK_\varepsilon]^{|X|} \prod_{e\in E} \beta J_e e^{\beta J_e} \\
&\le [eK_\varepsilon] \sum_{n\ge 1} \sum_{\gamma\in\mathcal{W}_a^{2n}} [e^{1+\beta} K_\varepsilon \beta]^n \prod_{j=0}^{2n-1} J_{x_j x_{j+1}}^{1/2} \\
&\le [eK_\varepsilon] \sum_{n\ge 1} [C\beta]^n = [eK_\varepsilon] \frac{C\beta}{1 - C\beta} < 1
\end{aligned}$$

where we used $|X| \leq |E| + 1$, $J_e \leq 1$ and we defined

$$C = e^{1+\beta} K_\varepsilon \left[\sup_x \sum_y J_{xy}^{1/2} \right]^2 .$$

The last line holds if β is small enough. The sum over paths is done starting from the end point

$$\sum_{\gamma \in \mathcal{W}_a^{2n}} \prod_{j=0}^{2n-1} J_{x_j x_{j+1}}^{1/2} = \sum_{x_1} \cdots \sum_{x_{2n}} \prod_{j=0}^{2n-1} J_{x_j x_{j+1}}^{1/2} \leq \left[\sup_x \sum_y J_{xy}^{1/2} \right]^{2n} .$$

This ends the proof. □

2.7 Polymer expansion

In this section we will prove Theorem 3. If we try to apply the same strategy as in the previous section, by Lemma 2 we can bound

$$V_c(X_1, \ldots, X_n) \leq \sum_{G \in \mathcal{G}_c[\{1,\ldots,n\}]} \prod_{e \in E} |[V_e - 1]|, \qquad \text{where } |[V_e - 1]| = 0, 1.$$

In the special case when $X_1 = X_2 = \cdots = X_n$ we have $|[V_e - 1]| = 1$ for any $e \in E_{\{1,\ldots n\}}$. Then

$$\sum_{G \in \mathcal{G}_c[\{1,\ldots,n\}]} \prod_{e \in E} |[V_e - 1]| = \text{card } \mathcal{G}_c[\{1, \ldots, n\}] = O(2^{n^2}).$$

On the other hand the contribution from $\sum_{X, a \in X} |\mathcal{A}(X)|^n$ is at best ρ^n for some $\rho < 1$. Then we have

$$\sum_{n \geq 1} \frac{\rho^n}{n!} 2^{n^2} = +\infty.$$

The solution to this problem is to partially resum the connected graphs in V_c. The resummation must garantee that:

1. we keep explicitely a set of lines ensuring the graph is connected (the minimal structure ensuring this is true is a tree);
2. all possible choices of additional lines are resummed (any additional line will create a cycle in the graph);
3. since the same graph can be obtained by different tree graphs by adding the necessary number of loop line we must ensure we do not count the same graph twice.

The advantage is that choosing a tree costs only a $n!$ factor instead of 2^{n^2}. The next subsection contains the tools needed to perform these operations rigorously.

2.7.1 Resummation of the connected diagrams.

Most of the material presented in this section is based on [2] and [1].

The smallest connected diagram on some abstract set of points Λ is a tree graph. Now let $P \in \mathcal{P}[\Lambda]$ be a partition of Λ. The smallest graph compatible with P (i.e. whose connected components are the elements in P) is a forest. Therefore a forest is the minimal amount of lines (information) we need to garantee that the elements in P are connected. The following formula allows to extract forests from sums over general graphs in a consistent way.

Theorem 5 (Brydges-Kennedy-Abdesselam-Rivasseau Forest formula) *For any finite abstract set of points Λ let $u = \{u_e\}_{e\in\mathcal{E}_\Lambda} \in [0,1]^{\mathcal{E}_\Lambda}$. For any function $f : [0,1]^{\mathcal{E}_\Lambda} \to \mathbb{C}$ with continuous first derivative in each u_e we have*

$$f(\mathbf{1}) = \sum_{F\in\mathcal{F}[\Lambda]} \int_{[0,1]^{|F|}} \prod_{e\in F} ds_e \left[\prod_{e\in F} \partial_{u_e}\right] f(u)_{|u(s_F)}$$

where $\mathbf{1} \equiv u_e = 1$ $\forall e \in \mathcal{E}_\Lambda$ and for any $e = (x,y)$

$$u_{xy}(s_F) = \begin{cases} 0 & \text{if } x,y \text{ not connected by } F \\ \inf_{e\in\gamma_{xy}(F)} s_e & \text{otherwise} \end{cases} \tag{2.7.15}$$

and $\gamma_{xy}(F)$ is the unique simple path made of edges in F connecting x to y.

Proof We will prove the analog formula for ordered forests

$$f(\mathbf{1}) = \sum_{n\geq 0} \sum_{\substack{F\in\mathcal{F}_o[\Lambda] \\ |F|=n}} \int_{\Delta_n} \prod_{j=1}^{|F|} ds_j \left[\prod_{j=1}^{n} \partial_{u_{e_j}}\right] f(u)_{|u(s_F)} \tag{2.7.16}$$

where $\mathcal{F}[\Lambda]$ is replaced by $\mathcal{F}_o[\Lambda]$ the ordered forests $F = \{e_1, \dots e_{|F|}\}$ and

$$\Delta_n = \{(s_1, s_2, \dots, s_n) \in [0,1]^n |\ 1 \geq s_1 \geq s_2 \geq \cdots \geq s_n \geq 0\}.$$

The sum over unordered forests can be obtained by remarking that

$$\sum_{orders} \int_{\Delta_n} = \int_{[0,1]^{|F|}}.$$

The formula (2.7.16) is proved by iteration on the number of edges in the forest. To encode the different steps it is convenient to introduce the function

$$\chi: \begin{array}{ll} \mathcal{P}[\Lambda] & \to [0,1]^{\mathcal{E}_\Lambda} \\ P & \to \chi_P \end{array} \quad \text{where} \quad \chi_P(e) = \chi_P(x,y) = \begin{cases} 1 & \text{if } \exists X \in P, x,y \in X \\ 0 & \text{otherwise} \end{cases}$$

and (x,y) are the two points attached to e. In the special case when the partition P contains only one set $P = \Lambda$ we have $\chi_\Lambda(e) = 1$ $\forall e \in \mathcal{E}_\Lambda$. Now let F_j $(j = 1, \dots, n = |F|)$ be the forest containing only the first j lines $(e_1, \dots, e_j)$

of F, and let $P_{F_j} \in \mathcal{P}[\Lambda]$ be the corresponding partition of Λ into connected components. Then $F_n = F$ and we have

$$u(s_{F_n}) = s_n(\chi_{P_{F_n}} - \chi_{P_{F_{n-1}}}) + s_{n-1}(\chi_{P_{F_{n-1}}} - \chi_{P_{F_{n-2}}}) + \cdots + s_1\chi_{P_{F_1}}.$$

To prove this identity let $\gamma_{xy}(F) = (l_{j_1}, \ldots, l_{j_p})$, $j_1 < j_2 < \cdots < j_p$ be the path in F connecting x to y (for any pair (x,y) connected by F). Then (x,y) are connected by F_j $\forall j \geq j_p$ but they are not connected by F_{j_p-1}. Therefore $\chi_{P_{F_j}}(x,y) = 0$ when $j < j_p$ and $\chi_{P_{F_j}}(x,y) = 1$ when $j \geq j_p$. Inserting this in the expression above we obtain $u(s_{F_n})(x,y) = s_{j_p} = \inf_{e \in \gamma_{xy}(F)} s_e$.

To set up the iteration we introduce for any forest $F_n = \{e_1, \ldots, e_n\}$, $n \geq 1$ the function

$$\begin{array}{rll} \tilde{u}_{n+1}: & \Delta_{n+1} & \to [0,1]^{\mathcal{E}_\Lambda} \\ & (s_1, \ldots, s_{n+1}) & \to \tilde{u}(s_1, \ldots, s_{n+1}) \end{array}$$

where

$$\tilde{u}(s_1, \ldots, s_{n+1}) = s_{n+1}(\chi_\Lambda - \chi_{P_{F_n}}) + u(s_{F_n}). \tag{2.7.17}$$

For the first iteration step we define

$$\begin{array}{rll} \tilde{u}_1: & [0,1] & \to [0,1]^{\mathcal{E}_\Lambda} \\ & s_1 & \to \tilde{u}(s_1) = s_1\chi_\Lambda. \end{array}$$

With these definitions $\tilde{u}$ satisfies

$$\tilde{u}_{n+1}(s_1, \ldots, s_n, s_n) = \tilde{u}_n(s_1, \ldots, s_n) \quad \forall n \geq 2. \tag{2.7.18}$$

Now we can start the proof of (2.7.16). Since $\mathbf{1} = \tilde{u}_1(s_1 = 1)$

$$f(\mathbf{1}) = f(\tilde{u}_1(s_1))_{|s_1=1} = f(\tilde{u}_1(s_1))_{|s_1=0} + \int_0^1 ds_1 \sum_{e_1 \in E_\Lambda} \partial_{u_{e_1}} f(u)_{|\tilde{u}_1(s_1)}$$

Using (2.7.15) $f(\tilde{u}_1(s_1))_{|s_1=0} = f(\mathbf{0}) = f(u(s_{F=\emptyset}))$ is the contribution from the unique forest with no line $|F| = 0$. Then the formula above can be written as

$$f(\mathbf{1}) = f(u(s_{F=\emptyset})) + \sum_{\substack{F_1 \in \mathcal{F}_o[\Lambda] \\ |F_1|=1}} \int_0^1 ds_1 \prod_{e \in F_1} \partial_{u_e} f(u)_{|\tilde{u}_1(s_1)}$$

To prove the induction step let us suppose

$$f(\mathbf{1}) = \left[\sum_{n=0}^{q-1} \sum_{\substack{F \in \mathcal{F}_o[\Lambda] \\ |F|=n}} \int_{\Delta_n} \prod_{j=1}^n ds_j \prod_{e \in F} \partial_{u_e} f(u)_{|u(s_F)} \right] + \left[\sum_{\substack{F_q \in \mathcal{F}_o[\Lambda] \\ |F|=q}} \int_{\Delta_q} \prod_{j=1}^q ds_j \prod_{e \in F_q} \partial_{u_e} f(u)_{|\tilde{u}_q(s_1,\ldots,s_q)} \right]$$

Using (2.7.18) (2.7.17)

$$\tilde{u}_q(s_1,\ldots,s_q) = \tilde{u}_{q+1}(s_1,\ldots,s_q,s_q), \qquad \tilde{u}_{q+1}(s_1,\ldots,s_q,0) = u(s_{F_q}),$$

hence

$$\prod_{e\in F_q} \partial_{u_e} f(u)_{|\tilde{u}_q(s_1,\ldots,s_q)} = \prod_{e\in F_q} \partial_{u_e} f(u)_{|u(s_{F_q})}$$
$$+ \int_0^{s_q} ds_{q+1} \sum_{e_{q+1}} \prod_{e\in F_{q+1}} \partial_{u_e} f(u)_{|\tilde{u}_{q+1}(s_1,\ldots,s_{q+1})}$$

where F_{q+1} is the forest with lines $e_1,\ldots e_{q+1}$. If $P_{F_{q+1}} = \Lambda$, i.e. the forest is a spanning tree on Λ then $\chi_\Lambda = \chi_{P_{F_{q+1}}}$,

$$\tilde{u}_{q+1}(s_1,\ldots,s_{q+1}) = u(s_{F_{q+1}})$$

and the induction stops. This ends the proof. □

In order to apply this formula to V_c we will need the following additional definition.

Definition: factorization. We say that a configuration $u \in [0,1]^{\mathcal{E}_\Lambda}$ is factorized on a partition $P \in \mathcal{P}[\Lambda]$ if $u_e = 0$ for all e edges connecting two differents sets in P:

$$u_{xy} = 0 \quad \text{if } \exists X \neq X' \in P \;, x \in X, y \in X'.$$

We say that a function $f : [0,1]^{\mathcal{E}_\Lambda} \to \mathbb{C}$ has the factorization property if for any partition $P \in \mathcal{P}[\Lambda]$ and any configuration u factorized on P

$$f(u) = \prod_{X\in P} f_X(u_X)$$

for some functions $f_X : [0,1]^{\mathcal{E}_X} \to \mathbb{C}$. Here u_X is the restriction of u to $\mathcal{E}_X$.

With this definition we have the following two lemmas.

Lemma 6 *Let $f : [0,1]^{\mathcal{E}_\Lambda} \to \mathbb{C}$ be a function with continuous first derivative in each u_e and with the factorization property. Then*

$$f(\mathbf{1}) = \sum_{P\in\mathcal{P}[\Lambda]} \prod_{X\in P} \left[\sum_{T\in\mathcal{T}[X]} \int_{[0,1]^{|T|}} \prod_{e\in T} ds_e \left[\prod_{e\in T} \partial_{u_e}\right] f_X(u_X)_{|u(s_T)} \right]$$

Proof Direct application of the definition of a forest as a partition $P \in \mathcal{P}[\Lambda]$ plus a tree graph inside each connected component $X \in P$. □

Lemma 7 *The connected function $V_c(1,\ldots,n)$ introduced in (2.4.11) can be written as*

$$V_c(X_1,\ldots,X_n) = \sum_{T\in\mathcal{T}[\{1,\ldots,n\}]} \int_{[0,1]^{|T|}} \prod_{e\in T} ds_e \left[\prod_{e\in T} [V_e - 1]\right] \prod_{e\notin T} [V_e(u_e(s_T))], \tag{2.7.19}$$

where

$$V_e(u_e) = [1 + u_e(V_e - 1)] \tag{2.7.20}$$

Proof Using (2.7.20) we define

$$f(u) = \prod_{e \in \mathcal{E}_{\{1,\ldots,n\}}} V_e(u_e).$$

This function satisfies $f(\mathbf{1}) = \prod_e V_e$ and $f(\mathbf{0}) = 1$. Moreover f is differentiable in all u_e and has the factorization property. Hence we can apply Lemma 6 above. Since

$$\left[\prod_{e\in T} \partial_{u_e}\right] f_X(u_X)_{|u(s_T)} = \prod_{e\in T}[V_e - 1] \prod_{e\notin T} V_e(u_e(s_T))$$

we obtain the result. □

2.7.2 Proof of Theorem 3

This proof is mostly based on [1]. We insert the formula (2.7.19) above inside (2.5.14)

$$\begin{aligned} |V_c(X_1,\ldots,X_n)| &\leq \sum_{T\in\mathcal{T}[\{1,\ldots,n\}]} \int_{[0,1]^{|T|}} \prod_{e\in T} ds_e \prod_{e\in T} |V_e - 1| \prod_{e\notin T} |V_e(u_e(s_T))| \\ &\leq \sum_{T\in\mathcal{T}[\{1,\ldots,n\}]} \prod_{e\in T} |V_e - 1| \end{aligned}$$

since

$$0 \leq [V_e(u_e(s_T))] = [1 + u_e(V_e - 1)] \leq 1.$$

Then (2.5.14) can be bounded by

$$\begin{aligned} &\sum_{\substack{X_1\subset\Lambda, |X_1|\geq 2,\\ a\in X_1}} \tfrac{|\mathcal{A}(X_1)|}{|X_1|} + \sum_{n\geq 2}\sum_{T\in\mathcal{T}[\{1,\ldots,n\}]} \tfrac{1}{(n-1)!} \sum_{\substack{X_1\subset\Lambda, |X_1|\geq 2\\ a\in X_1}} \tfrac{|\mathcal{A}(X_1)|}{|X_1|}. \\ &\qquad \sum_{\substack{X_2,\ldots,X_n\subset\Lambda\\ |X_i|\geq 2\forall i}} \left[\prod_{i=2}^{n} |\mathcal{A}(X_i)|\right] \prod_{e\in T} |V_e - 1| \end{aligned} \tag{2.7.21}$$

In order to fix a tree $T \in \mathcal{T}[\{1,\ldots,n\}]$ we need three ingredients:

1. $d_1,\ldots,d_n \geq 1$ the coordination number of each vertex $i = 1,\ldots n$ (i.e. number of lines in T hooking to i); the choice of these numbers must be compatible with a tree structure;

2. a tree with fixed coordination numbers $d_1\ldots,d_n$, $T \in \mathcal{T}_{d_1,\ldots,d_n}[\{1,\ldots,n\}]$.

The sum over trees above can be reorganized as

$$\sum_T (\cdot) = \sum_{\{d_j\}_{j=1}^n} \sum_{T,\{d_j\}_{j=1}^n fixed} (\cdot) \leq \sum_{\{d_j\}_{j=1}^n} \tfrac{(n-2)!}{\prod_{j=1}^n (d_j-1)!} \sup_{T,\{d_j\}_{j=1}^n fixed} (\cdot).$$

where we used Cayley's theorem

$$\operatorname{card}[T,\{d_j\}_{j=1}^n fixed] = \frac{(n-2)!}{\prod_{j=1}^n (d_j-1)!}.$$

Then the terms corresponding to $n \geq 2$ in (2.7.21) are bounded by

$$\sum_{n\geq 2}\sum_{d_1,\ldots d_n} \sup_{T,\{d_j\}_{j=1}^n fixed} \sum_{\substack{X_1,\ldots,X_n\subset\Lambda \\ |X_i|\geq 2,\ a\in X_1}} \tfrac{1}{|X_1|}\left[\prod_{i=1}^n \tfrac{|\mathcal{A}(X_i)|}{(d_j-1)!}\right]\prod_{e\in T}|V_e-1|$$

For a fixed tree we call X_1 the root and all points with $d_j = 1$ (except eventually the root) are called leaves. For each point j there is a unique path in T going from j to the root. Note that the choice of the root is arbitrary. Therefore we can perform the sum over X_j recursively starting from the leaves and going towards the root. For any fixed point j let $e_j = (X_j, X_{a(j)})$ the unique edge in the tree going from j towards the root. We call $a(j)$ the ancestor of j in the tree.

When j is a leaf e_j is the unique edge touching j. Therefore X_j appears only in $\mathcal{A}(X_i)$ and in V_{e_j}. Keeping $X_{a(j)}$ fixed

$$\begin{aligned}\sum_{X_j}|\mathcal{A}(X_j)||1-V_{e_j}| &= \sum_{X_j}|\mathcal{A}(X_j)||1-V(X_j,X_{a(j)})| = \sum_{X_j,X_j\cap X_{a(j)}\neq\emptyset}|\mathcal{A}(X_j)| \\ &\leq |X_{a(j)}|\sup_{b\in X_{a(j)}}\sum_{X_j,b\in X_j}|\mathcal{A}(X_j)| \ \leq |X_{a(j)}|\sup_{b\in\Lambda}\sum_{X_j,b\in X_j}|\mathcal{A}(X_j)||X_j|^{d_j-1}\end{aligned}$$

where we used $d_j - 1 = 0$ since j is a leaf. After completing the sum for each leaf we pass to the next point $j' = a(j)$. Since each leaf attached to j' brings a factor $|X_{j'}|$ we will have to estimate

$$\begin{aligned}\sum_{X_{j'}}|\mathcal{A}(X_{j'})||X_{j'}|^{d_{j'}-1}|1-V_{e_{j'}}| &= \sum_{X_{j'},X_{j'}\cap X_{a(j')}\neq\emptyset}|\mathcal{A}(X_{j'})||X_{j'}|^{d_{j'}-1} \\ &\leq |X_{a(j')}|\sup_{b\in\Lambda}\sum_{X_{j'},b\in X_{j'}}|\mathcal{A}(X_{j'})||X_{j'}|^{d_j-1}.\end{aligned}$$

We repeat this operation for every vertex except the root. Finally for the root we have

$$\sum_{X_1,a\in X_1}|\mathcal{A}(X_1)||X_1|^{d_1-1} \leq \sup_{b\in\Lambda}\sum_{X_1,b\in X_1}|\mathcal{A}(X_1)||X_1|^{d_1-1}$$

Note that these bounds now depend on $d_1,\ldots,d_n$, but not on the specific tree, so the $\sup_T$ is trivial.

Then (2.7.21) is bounded by

$$\sum_{n\geq 1}\rho^n = \frac{\rho}{1-\rho} < \infty,$$

where we used

$$\sum_{d_j \geq 1} \sup_{b \in \Lambda} \sum_{X_j \subset \Lambda, b \in X_j} |\mathcal{A}(X_j)| \tfrac{|X|^{d_j - 1}}{(d_j - 1)!} = \sup_{b \in \Lambda} \sum_{X_j \subset \Lambda, b \in X_j} |\mathcal{A}(X_j)| e^{|X_j|} \leq \rho < 1$$

uniformly in Λ by Theorem2 and

$$\sup_{b \in \Lambda} \sum_{X_1 \subset \Lambda, b \in X_1} \frac{|\mathcal{A}(X)|}{|X_1|} \leq \rho$$

for the contribution from $n = 1$. This ends the proof. □

2.7.3 Proof of Theorem 4

Using the results above we can write

$$\begin{aligned}
\tfrac{1}{|\Lambda|} \left| \ln \tfrac{Z_{\Lambda,\beta}(\mathbf{h}_{|\Lambda})}{Z_{\Lambda,0}(\mathbf{h}_{|\Lambda})} \right| &\leq \tfrac{1}{|\Lambda|} \sum_{n \geq 1} \tfrac{1}{n!} \sum_{\substack{X_1, \ldots, X_n \subset \Lambda \\ |X_i| \geq 2,\ \forall i}} \left[\prod_{i=1}^{n} |\mathcal{A}(X_i)| \right] |V_c(X_1, \ldots, X_n)| \\
&\leq \tfrac{1}{|\Lambda|} \sum_{a \in \Lambda} \sum_{n \geq 1} \tfrac{1}{(n-1)!} \sum_{\substack{X_1, \ldots, X_n \subset \Lambda, a \in X_1 \\ |X_i| \geq 2,\ \forall i}} \tfrac{1}{|X_1|} \left[\prod_{i=1}^{n} |\mathcal{A}(X_i)| \right] |V_c(X_1, \ldots, X_n)| \\
&\leq \sup_{a \in \Lambda} \sum_{n \geq 1} \tfrac{1}{(n-1)!} \sum_{\substack{X_1, \ldots, X_n \subset \Lambda, a \in X_1 \\ |X_i| \geq 2,\ \forall i}} \tfrac{1}{|X_1|} \left[\prod_{i=1}^{n} |\mathcal{A}(X_i)| \right] |V_c(X_1, \ldots, X_n)| < \infty
\end{aligned}$$

where in the second line we used

$$\begin{aligned}
\sum_{\substack{X_1, \ldots, X_n \subset \Lambda \\ |X_i| \geq 2,\ \forall i}} (\cdot) = \sum_{\substack{X_1, \ldots, X_n \subset \Lambda \\ |X_i| \geq 2,\ \forall i}} \tfrac{1}{|\cup_j X_j|} \sum_{a \in \cup_j X_j} (\cdot) &= \sum_{a \in \Lambda} \sum_{\substack{X_1, \ldots, X_n \subset \Lambda, a \in \cup_j X_j \\ |X_i| \geq 2,\ \forall i}} \tfrac{1}{|\cup_j X_j|} (\cdot) \\
&\leq n \sum_{a \in \Lambda} \sup_{j} \sum_{\substack{X_1, \ldots, X_n \subset \Lambda, a \in X_j \\ |X_i| \geq 2,\ \forall i}} \tfrac{1}{|X_j|} (\cdot)
\end{aligned}$$

and without loss of generality we fixed $j = 1$. This ends the proof. □

2.8 Cluster expansion: general case

With the forest formulas introduced above we can now generalize Theorem 2 to any interaction J_{xy} such that $\sum_y J_{xy} = 1$. For this purpose we resum partially the connected graphs in w_c. Using the tree formula we have

$$\begin{aligned}
w_c(X) &= \sum_{T \in \mathcal{T}[X]} \int_{[0,1]^{|T|}} \prod_{e \in T} ds_e \prod_{e \in T} \beta J_e |(S_{x_e}, S_{y_e})| \prod_{e \in \mathcal{E}_X} |w_e(u_e(s_T))| \\
&\leq \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} \beta J_e \prod_{e \in \mathcal{E}_X} e^{\beta J_e}
\end{aligned}$$

where we defined $e = (x_e, y_e)$,

$$w_e(u) = e^{\beta u_e J_e(S_{x_e}, S_{y_e})}$$

and we used $|(S_{x_e}, S_{y_e})| \leq 1$ and

$$0 \leq e^{\beta u_e J_e(S_{x_e}, S_{y_e})} \leq e^{\beta J_e}.$$

Since $\sum_y J_{xy} = 1$ we have

$$\prod_{e \in \mathcal{E}_X} e^{\beta J_e} \leq e^{\beta |X|}.$$

Then

$$\sum_{\substack{X \subset \Lambda, a \in \Lambda \\ |X| \geq 2}} \mathcal{A}(X) e^{|X|} \leq \sum_{\substack{X \subset \Lambda, a \in \Lambda \\ |X| \geq 2}} K^{|X|} \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} \beta J_e = K \sum_{\substack{X \subset \Lambda, a \in \Lambda \\ |X| \geq 2}} \sum_{T \in \mathcal{T}[X]} \prod_{e \in T} [\beta K J_e] \tag{2.8.22}$$

where $K = e^{1+\beta}(1 + O(\varepsilon))$ where we used $|T| = |X| - 1$. For any given $a \in Y$ and any $Y \subset \Lambda$ we define

$$H_{Y,a} = \sum_{\substack{X \subset Y, a \in Y \\ |Y| \geq 1}} \sum_{T \in \mathcal{T}[Y]} \prod_{e \in T} [\beta K J_e]$$

With this definition (2.8.22) corresponds to $H_{\Lambda,a} - 1$, since we consider only subsets of Λ with $|X| \geq 2$. Now $H_{Y,a}$ satisfies the relation

$$H_{Y,a} = \sum_{d=0}^{|Y|-1} \frac{1}{d!} \sum_{x_1, \ldots x_d \in Y} \sum_{\substack{X_1, \ldots, X_d \subset Y \setminus \{a\} \\ x_i \in X_j \forall j}} \prod_{j=1}^{d} \left[\beta K J_{a,x_j} \sum_{T \in \mathcal{T}[X_j]} \prod_{e \in T} [\beta K J_e] \right] V(X_1, \ldots, X_d)$$

where the subset X_j must contain the point x_j, the potential $V(X_1, \ldots, X_d)$ ensures the sets X_j do not overlap and d is the coordination number of a in the tree. Finally we have the condition $X_j \subset Y \setminus \{a\}$ since a already belongs to the tree. By neglecting the interaction $0 \geq V \geq 1$ we obtain

$$H_{Y,a} \leq \sum_{d \geq 0} \frac{1}{d!} \sum_{x_1, \ldots x_d \in Y} \prod_{j=1}^{d} \left[\beta K J_{a,x_j} H_{Y \setminus \{a\}, x_j} \right]$$

We will prove by induction on the size of Y that

$$H_{Y,a} \leq e^{\sqrt{\beta}} \qquad \forall Y$$

Indeed when $|Y| = 1$ then $H_{Y,a} = 1 \leq e^{\sqrt{\beta}}$. Let us suppose the bound is true for $|Y| \leq m$. Let $|Y| = m + 1$. Applying the estimate above

$$H_{Y,a} \leq \sum_{d \geq 0} \frac{1}{d!} \sum_{x_1, \ldots x_d \in Y} \prod_{j=1}^{d} [\beta K J_{a,x_j} H_{Y \setminus \{a\}, x_j}] \leq \sum_{d \geq 0} \frac{1}{d!} [\beta K e^{\sqrt{\beta}}]^d = e^{\beta K e^{\sqrt{\beta}}} < e^{\sqrt{\beta}}$$

for β small enough. Here we used $|Y\setminus\{a\}| = m$ and $\sum_y J_{xy} = 1$. Finally

$$K \sum_{\substack{X\subset\Lambda, a\in\Lambda \\ |X|\geq 2}} \sum_{T\in\mathcal{T}[X]} \prod_{e\in T} [\beta K J_e] = K\,[H_{\Lambda,a} - 1] \leq K\left[e^{\sqrt{\beta}} - 1\right] < 1$$

for β small enough. This ends the proof. $\square$

Bibliography

[1] D. Brydges, *"A short course on cluster expansions"*,

Les Houches 1984

[2] A. Abdesselam, V. Rivasseau *"Trees, forests and jungles: A botanical garden for cluster expansions "* ,

Constructive Physics Results in Field Theory, Statistical Mechanics and Condensed Matter Physics Lecture Notes in Physics Volume 446, 1995, pp 7-36

[3] S. Friedli, Y. Velenik *"Equilibrium Statistical mechanics of classical lattice systems: a concrete introduction."*,

http://www.unige.ch/math/folks/velenik/smbook/index.html