Processes with reinforcement

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Joint work with
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Vertex-reinforced jump process (VRJP)

- Let $\mathcal{G} = (V, E)$ be a locally finite, undirected graph without direct loops. The undirected edge $i \sim j$ is given the weight $W_{ij} > 0$.
- The vertex-reinforced jump process $Y = (Y_u)_{u \geq 0}$ is a process in continuous time where given $(Y_s)_{s \leq u}$ the particle jumps from site i to site $j \sim i$ with rate

$$W_{ij}(1 + L_j(u)),$$

where

$$L_j(u) = \int_0^u 1_{\{Y_s = j\}} \, ds$$

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- Conceived by Wendelin Werner.
- [Davis-Volkov 2002,2004]: finite graphs and trees.
- [Collevecchio 2009], [Basdevant-Singh 2010]: more on trees

An important discovery

Let

$$\tilde{Y} = (\tilde{Y}_n)_{n\in\mathbb{N}_0}$$

be the discrete-time process associated with the VRJP Y by taking only the value of Y_u immediately before the jump times.

Theorem (Sabot-Tarrès 2011)

On any finite graph, $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ is a mixture of reversible Markov chains.

Mixture of reversible Markov chains

On any finite graph, the discrete-time process $\tilde{Y}$ associated with the VRJP is a mixture of reversible Markov chains. This means:

- Set $\mathcal{X} = (0, \infty)^E$.
- $\mathcal{X} \ni x = (x_{ij})_{(i \sim j) \in E}$ are weights on the undirected edges of the graph G.

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- For $i_0 \in V$ and $x \in (0, \infty)^E$, let $Q^{\mathrm{mc}}_{i_0, x}$ be the distribution of the discrete-time Markovian random walk, starting at i_0, with transition matrix

$$\Pi_x(i, j) = \frac{x_{ij}}{\sum_{k \in V : i \sim k} x_{ik}} 1_{\{(i \sim j) \in E\}}.$$

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There is a unique probability measure $\mathbb{P}^W_{i_0}$ on $\mathcal{X}$, depending on the starting point i_0 and the weights $W = (W_{ij})_{(i\sim j)\in E} \in \mathcal{X}$ of the VRJP such that for any event $A \subseteq V^{\mathbb{N}_0}$, one has

$$P^{W,\mathrm{vrjp}}_{i_0}(\tilde{Y} \in A) = \int_{\mathcal{X}} Q^{\mathrm{mc}}_{i_0,x}(A)\, \mathbb{P}^W_{i_0}(dx).$$

Description of the mixing measure

Theorem (Sabot-Tarrès 2011)

The mixing measure $\mathbb{P}_{i_0}^W$ can be described by putting on the edge $i \sim j$ the weight

$$W_{ij} e^{u_i + u_j}$$

with $(u_i)_{i \in V}$ distributed according to (a marginal of) Zirnbauer's supersymmetric (susy) hyperbolic non-linear sigma model.

- introduced by Zirnbauer in 1991.
- Zirnbauer writes that it may serve as a toy model for studying diffusion and localization in disordered one-electron systems.
- Original form involves Grassmann variables.
- Present version: Grassmann variables replaced by spanning trees.

The supersymmetric hyperbolic non-linear sigma model

$$\text{Let} \quad \Omega_{i_0} = \{(u,s) \in \mathbb{R}^V \times \mathbb{R}^V : u_{i_0} = s_{i_0} = 0\},$$
$$\mathcal{T} = \text{set of spanning trees of } \mathcal{G}.$$

$\mu_{i_0}^W$ is the following probability measure on $\Omega_{i_0} \times \mathcal{T}$:

$$d\mu_{i_0}^W(u,s,T)$$
$$= \prod_{(i\sim j)\in E} \exp\left\{-W_{ij}\left[\cosh(u_i - u_j) - 1 + \frac{1}{2}(s_i - s_j)^2 e^{u_i+u_j}\right]\right\}$$
$$\prod_{(i\sim j)\in T} W_{ij} e^{u_i+u_j} \prod_{j\in V\setminus\{i_0\}} \frac{e^{-u_j}\, du_j ds_j}{2\pi}\, dT,$$

where du_j and ds_j denote the Lebesgue measure on $\mathbb{R}$ and dT is the counting measure on $\mathcal{T}$.

Interpretation of the supersymmetric sigma model

$$(c(u)W_{ij}e^{u_i+u_j})_{(i\sim j)\in E}$$

with $c(u) = 1/(\sum_{(i\sim j)\in E} W_{ij}e^{u_i+u_j})$ can be interpreted as the asymptotic fraction of time the discrete time process associated to the vertex-reinforced jump process spends traversing the edges of the graph $\mathcal{G}$.

Interpretation of the supersymmetric sigma model

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In a work in progress with Franz Merkl and Pierre Tarrès, we give an interpretation of the other variables s and T in terms of the asymptotic behavior of a time-changed version of the vertex-reinforced jump process.

- T corresponds to the last exit tree,
- s corresponds to fluctuations of the local times spent at the vertices around their limit.

Even more, we provide an extension of the susy model having $\mu_{i_0}^W$ as a marginal.

Key results on the supersymmetric sigma model

The following results were shown for the susy model on $\mathbb{Z}^d$ with constant $W_{ij} \equiv W$:

Exponential localization

- for $d = 1$ and all W
  [Zirnbauer 1991] and [Disertori-Spencer 2010]
- for $d \geq 2$ and small W
  [Disertori-Spencer 2010]

Quasi-diffusive phase

- for $d \geq 3$ and large W
  [Disertori-Spencer-Zirnbauer 2010]

Recurrence and transience on $\mathbb{Z}^d$

Consider the graph $\mathbb{Z}^d$ with constant weights W.

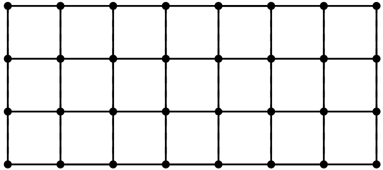
The previous connections were used to prove a phase transition for the vertex-reinforced jump process for $d \geq 3$:

- [Sabot-Tarrès 2011]
  recurrence for $d \geq 2$ for small weights W
- [Disertori-Sabot-Tarrès 2014]
  transience for $d \geq 3$ and large weights W

The supersymmetric sigma model on a strip

- Consider $\mathcal{G} = \mathbb{Z} \times G$ with a finite connected undirected graph G.
- For $L = (\underline{L}, \overline{L}) \in \mathbb{N}^2$ consider large finite pieces

$$\mathcal{G}_L = \{-\underline{L}, \overline{L}\} \times G$$



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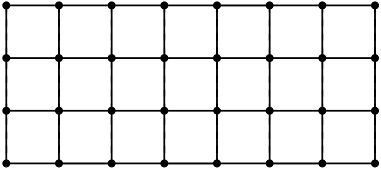

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- Endow $\mathcal{G}$ with translation-invariant edge weights W_{ij}.
- Fix one point $\mathbf{0} = (0, p)$ at level 0.
- Extend $\mathcal{G}$ and $\mathcal{G}_L$ by adding a vertex ρ which is only connected to $\mathbf{0}$. Denote the new graphs by $\mathcal{G}^\rho$ and $\mathcal{G}_L^\rho$.
- Let $\mu_{\mathbf{0},L}^W$ denote the susy measure corresponding to $\mathcal{G}_L^\rho$.

Exponential localization

For $l \in \mathbb{Z}$, let $\ell := (l, p)$ denote the copy of $\mathbf{0} = (0, p)$ at level l.

Theorem (Disertori, Merkl & R. 2014)

For all $L = (-\underline{L}, \overline{L})$ and l with $-\underline{L} \leq l \leq \overline{L}$, one has

$$E_{\mu^W_{\mathbf{0},L}}\left[e^{\frac{u_\ell - u_\mathbf{0}}{2}}\right] \leq c_1 e^{-c_2|l|}$$

with constants $c_1(G, W), c_2(G, W) > 0$.

Existence of an infinite volume limit

Theorem (Disertori, Merkl & R. 2014)

There exists a probability measure

$$\mu^W_{\mathbf{0},\infty} \text{ on } \mathbb{R}^V \times \mathbb{R}^V$$

such that for any bounded random variable $\mathcal{O}$ depending only on finitely many u_i, s_i we have

$$E_{\mu^W_{\mathbf{0},L}}[\mathcal{O}] \to E_{\mu^W_{\mathbf{0},\infty}}[\mathcal{O}] \qquad \text{as } L = (-\underline{L}, \overline{L}) \to (-\infty, +\infty).$$

Corollary 1 for VRJP on the infinite graph $\mathcal{G}^\rho$

Using the result from Sabot and Tarrès, our exponential decay for the sigma model has the following consequences for the vertex-reinforced jump process (VRJP):

Corollary (Disertori, Merkl & R. 2014)

The discrete time process associated to the VRJP on $\mathcal{G}^\rho$ is a mixture of positive recurrent irreducible reversible Markov chains.

The mixing measure for the random weights is given by the joint distribution of

$$\left(W_{ij} e^{u_i + u_j}\right)_{(i \sim j) \in \mathcal{G}^\rho}$$

with respect to $\mu^W_{\mathbf{0},\infty}$.

Corollary 2 for VRJP on the infinite graph $\mathcal{G}^\rho$

Corollary (Disertori, Merkl & R. 2014)

For the discrete-time process $(\tilde{Y}_n)_{n\in\mathbb{N}_0}$ associated to the VRJP on $\mathcal{G}^\rho$, one has

$$\sup_{n\in\mathbb{N}_0} P_\rho^{W,\mathrm{vrjp}}(\tilde{Y}_n = i) \le c_3 e^{-c_4|i|} \quad \textit{for all } i \in V,$$

$$\max_{k=0,\ldots,n} |\tilde{Y}_k| \le \; c_5 \log n \quad \textit{for all large } n \qquad P_\rho^{W,\mathrm{vrjp}}\textit{-a.s.}$$

with constants $c_3(G,W), c_4(G,W), c_5(G,W) > 0$.

Ideas from the proof

Key estimate: $E_{\mu^W_{\mathbf{0},L}}\left[e^{\frac{t_\ell - t_{\mathbf{0}}}{2}}\right] \leq c_1 e^{-c_2 l}$, where

$$\begin{aligned} & d\mu^W_{\mathbf{0},L}(u,s,T) \\ = & \prod_{(i\sim j)\in E} \exp\left\{-W_{ij}\left[\cosh(u_i-u_j)-1+\frac{1}{2}(s_i-s_j)^2 e^{u_i+u_j}\right]\right\} \\ & \prod_{(i\sim j)\in T} W_{ij}e^{u_i+u_j}\cdot W_{\mathbf{0}\rho}e^{u_{\mathbf{0}}}e^{-M(u_{\mathbf{0}},s_{\mathbf{0}})}\prod_{j\in V}\frac{e^{-u_j}\,du_j ds_j}{2\pi}\,dT \end{aligned}$$

Strategy of the proof:

- Use a transfer operator approach.
- We need a local description of the spanning trees for every "slice" of the graph.
- The proof has some similarities with the first recurrence proof for linearly edge-reinforced random walk on $\mathbb{Z}\times\{1,2\}$ [Merkl & R. 2005].

Linearly edge-reinforced random walk (ERRW)

Fix weights $a_{ij} > 0$, $(i \sim j) \in E$.
Linearly edge-reinforced random walk is a discrete-time process $(X_n)_{n\in\mathbb{N}_0}$ on $\mathcal{G}$ starting in i_0.

The reinforcement is encoded in time-dependent weights $w_{ij}(n)$, $n \in \mathbb{N}_0$, on the undirected edges $(i \sim j) \in E$.

- Initial weights: $w_{ij}(0) = a_{ij}$
- Each time an edge is crossed, its weight is increased by 1:

$$w_{ij}(n+1) = w_{ij}(n) + 1_{\{(X_n \sim X_{n+1}) = (i \sim j)\}}.$$

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The process jumps with probability proportional to the edge weights: For $j \in V$, $n \in \mathbb{N}_0$,

$$P_{i_0}^{a,\mathrm{errw}}(X_{n+1} = j | X_0, \ldots, X_n) = \frac{w_{X_n j}(n)}{\sum_{(X_n \sim k) \in E} w_{X_n k}(n)} 1_{\{(X_n \sim j) \in E\}}.$$

ERRW as a mixture of Markov chains

- ERRW was introduced by Diaconis in 1986.
- Pemantle in his thesis writes that Diaconis asked him about recurrence and transience of the process on $\mathbb{Z}^d$, $d \geq 1$. [Pemantle, 1988] showed a phase transition between recurrence and transience on a binary tree.

It has been known for a long time that ERRW on any finite graph is a mixture of reversible Markov chains. The mixing measure can be described as a joint probability law on the set $(0, \infty)^E$ of edge weights of the graph.

- ERRW is partially exchangeable. Hence one can apply de Finetti theorems: [Diaconis-Freedman, 1980], [R., 2003]
- "The magic formula": [Coppersmith-Diaconis, 1986], [Keane-R., 2000], [Merkl-Öry-R., 2008], [Sabot-Tarrès-Zeng, 2016], ...

Results on ERRW

Using the explicit description of the mixing measure, many results on linearly edge-reinforced random walks were proved, among others, recurrence and asymptotic properties of the process

- for $\mathbb{Z} \times G$ with a finite graph G and arbitrary constant initial weights [Merkl & R., 2005-2009],
- for a diluted version of $\mathbb{Z}^2$ with small initial weights [Merkl & R., 2009].

In [Merkl & R., 2008], we proved polynomial decay of the edge weights for $\mathbb{Z}^2$. However, to deduce recurrence, we needed fast enough decay which we could only prove for small initial weights and a dilution of $\mathbb{Z}^2$.

Methods:

- transfer operator
- symmetry for finite pieces with periodic boundary conditions

A connection with the supersymmetric sigma model

Theorem (Sabot-Tarrès 2011)

The law of linearly edge-reinforced random walk X is a mixture of the law of the discrete-time process $\tilde{Y}$ associated to VRJP if one takes W_{ij} independent Gamma(a_{ij})-distributed.

Let $\Gamma_{a_{ij}}$ denote the gamma distribution with parameter a_{ij}. Then, for any event $A \subseteq V^{\mathbb{N}_0}$, one has

$$P_{i_0}^{a,\mathrm{errw}}(X \in A) = \int_{(0,\infty)^E} P_{i_0}^{W,\mathrm{vrjp}}(\tilde{Y} \in A) \prod_{W_{ij} \in E} \Gamma_{a_{ij}}(dW_{ij})$$

$$= \int_{(0,\infty)^E} \int_{\Omega_{i_0}} Q^{\mathrm{mc}}_{i_0, W_{ij} e^{u_i + u_j}}(A)\, \mu_{i_0}^{W}(du) \prod_{W_{ij} \in E} \Gamma_{a_{ij}}(dW_{ij}).$$

Consequences for ERRW

This connection allowed to transfer results from the susy model to ERRW. Consider ERRW on $\mathbb{Z}^d$ with constant initial weights.

- [Sabot-Tarrès 2011]
  recurrence for $d \geq 2$ for small initial weights
- [Disertori-Sabot-Tarrès 2014]
  transience for $d \geq 3$ and large initial weights

[Angel-Crawford-Kozma 2012]
gave an alternative proof for the recurrence part without using the connection to the non-linear supersymmetric sigma model.

Recurrence of ERRW on $\mathbb{Z}^2$

Theorem (Sabot-Zeng 2015)

On $\mathbb{Z}^2$, linearly edge-reinforced random walk is recurrent for all constant initial weights.

Ideas of the proof of [Sabot-Zeng, 2015]:

- Consider boxes $V_n = [-n, n]^2$ with wired boundary conditions. One can couple the corresponding susy models. This yields coupled variables $u_i^{(n)}, i \in V_n$.
- $e^{u_i^{(n)}}, n \in \mathbb{N}$, is a martingale.
  Extension by [Disertori, Merkl & R., 2015] to a hierarchy of martingales.
- The VRJP is transient iff $\lim_{n\to\infty} e^{u_i^{(n)}} > 0$.
- Using polynomial decay of the edge weights describing the mixing measure for ERRW from [Merkl & R., 2008], they deduce recurrence for $\mathbb{Z}^2$.

Summary

The supersymmetric sigma model, originally designed as a toy model for disordered media, unexpectedly provides a powerful tool to study

- vertex-reinforced jump processes and
- linearly edge-reinforced random walk.