Exponential approach to equilibrium for a stochastic NLS

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I. Stochastic NLS

We start with the dispersive nonlinear Schrödinger equation (NLS):

$$-i\dot{u} = -\Delta u \pm |u|^{p-2}u,$$

on the torus; u is a function of time and the space variable on the torus: $u = u(t, x)$; plus sign is called defocusing and minus focusing. (As a memory device, both terms on the RHS are of the same sign in defocusing; while of opposite sign in focusing.) We specialize to the cubic NLS on the circle, i.e., $p = 4$, and u may be identified with a periodic function:

$$u(\cdot, x) = u(\cdot, x + 2\pi).$$

The NLS is a Hamiltonian equation with the Hamiltonian

$$H(u, \bar{u}) = \frac{1}{2} \int (|\nabla u|^2) \pm \frac{1}{4}|u|^4,$$

and u and $\bar{u}$ are conjugate variables. It has two conserved quantities, namely the energy (Hamiltonian) and the mass, the L^2 norm of the solution u.

For our purposes, it is more convenient to consider a variant of the above NLS:

$$-i\dot{u} = -\Delta u \pm |u|^2 u + \|u\|_2^{2r} u.$$

Since the L^2 norm is a conserved quantity, this does not change the nature of the problem.

Below we look at the focusing case. (The defocusing case is simpler and the same results hold.)

Further, we consider a stochastic version of it, namely the following (damped) SNLS:

$$du + (\nu\sigma * \sigma * + i)(-\Delta + 1 - \lambda|u|^2 + \|u\|_2^{2r})u\,dt = \sqrt{2\nu}\sigma * dW,$$

where σ is a smoothing operator, which damps high Fourier modes, $\nu, \lambda > 0$ are parameters and W is a white in space and time complex Wiener process. (The mass term 1 was added to avoid the zero mode.)

The above SNLS is chosen in order that e^{-H} is an invariant measure. More generally $e^{-\beta H}$ is an invariant measure after replacing ν by $\nu\beta$, with β the inverse temperature.

Note that unlike the dispersive case, H is not a conserved quantity due to dissipation. So $e^{-\beta H}$ is not automatically invariant. One needs to choose the "right" process for it to be invariant.

Below I shall describe a series of recent and ongoing collaborative works among Carlen, Fröhlich, Lebowitz and myself.

That $\mu = e^{-\beta H}$ is well-defined is shown in a recent work of Carlen, Fröhlich and Lebowitz (2015). Note that previously, it was shown in a work of Lebowitz, Rosen and Speer that without the $\|u\|_2^{2r}$ term, the measure $e^{-\beta H}$ is well-defined in an L^2-ball.

The addition of the term $\|u\|_2^{2r}$ avoids the need for boundary conditions and is technically simpler, cf. Lebowitz, Mounaix and Wang (2013) for the case with boundary condition. (Recall that contrary to the dispersive NLS, for SNLS the L^2 norm is generally not conserved, which is why boundary conditions are needed.)

Remark. The invariant measure is independent of σ, which only affects the dynamics.

II. Fokker-Planck equation in the Fourier variables

In the Fourier variables, the Hamiltonian becomes a sum denoted by Φ:

$$\Phi(a)/2\pi := \Phi(a, \bar{a})/2\pi$$
$$= \sum_{n\in\mathbb{Z}} (n^2+1)|a_n|^2 - \lambda \sum_{n_1-n_2+n_3-n_4=0} a_{n_1}\bar{a}_{n_2}a_{n_3}\bar{a}_{n_4} + (\sum_n |a_n|^2)^r,$$

where $a_n \in \mathbb{C}$, n^2 stands for $|n|^2$.

We have the following equation for a:

$$\dot{a}_n = -i\frac{\partial \Phi}{\partial \bar{a}_n} - A_n\frac{\partial \Phi}{\partial \bar{a}_n} + \sqrt{A_n}\Gamma_n, \qquad |n| \leq N,$$

where the Γ_n are independent centered complex Gaussian white noises with

$$\langle \Gamma_n(t)\bar{\Gamma}_n(t')\rangle = 2\delta(t-t');$$

diag A_n is the operator $\sigma * \sigma = (-\Delta + 1)^{-\gamma}$, $\gamma > 0$ in Fourier space.

We will not directly deal with the PDE, but instead the associated Fokker - Planck equation which describes the time evolution of the probability density $\tilde{P}(a,t)$ with initial distribution $\tilde{P}(a,0)$ for the process given by the stochastic PDE.

Conjugating by $e^{\pm\Phi}$ and setting $P = e^{\Phi}\tilde{P}$, the Fokker - Planck equation has the form:

$$\partial_t P(a,t) + \mathcal{L}P(a,t) = 0,$$

where

$$\mathcal{L} = -\sum_n A_n \frac{\partial^2}{\partial \bar{a}_n \partial a_n} + \langle A\nabla\Phi, \nabla\Phi\rangle - \sum_n A_n \frac{\partial^2}{\partial \bar{a}_n \partial a_n}\Phi + \mathcal{H}_\Phi.$$

Here

$$\nabla\Phi \text{ is the vector } \{\partial_{\bar{a}_n}\Phi, \partial_{a_n}\Phi\},$$

and

$$\mathcal{H}_\Phi = 2\sum_n (\partial_{a_n}\Phi\partial_{\bar{a}_n} - \partial_{\bar{a}_n}\Phi\partial_{a_n})$$

is the Hamiltonian vector field.

There is the following result:

Theorem. Let $P_0\mu$ be the initial distribution, then the distribution at time t satisfies

$$\|P_t - 1\|_{L^2(\mu)} \leq e^{-tE}\|P_0 - 1\|_{L^2(\mu)};$$

with

$$E \geq 1 - \big(\frac{\lambda}{1-\frac{2}{r}}\big)^{\frac{r}{r-1}}\big(\frac{r-1}{r}\big)\frac{1}{(r)^{\frac{1}{r-1}}}$$

$$\sim 1 - \lambda,$$

for $r \gg 1$. So $E > 0$ for $\lambda < 1$.

Remark. The conditions for $E > 0$ correspond to the Hamiltonian Φ being convex. One makes Fourier truncation and estimates uniform in dimension. The convexity condition can be improved by using Holley-Strook and entropy.

IV. Heat equation on the cotangent space

Assume that f and g are two (non-negative) real solutions to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = \mathcal{L}f;$$

$$\frac{\partial g}{\partial t} = \mathcal{L}g.$$

We are mainly interested in the time evolution of inner products:

$$\frac{\partial}{\partial t} \int fgd\mu.$$

For this purpose, the problem simplifies – only the real part of the generator $\mathcal{L}$ contributes leading to a usual heat equation with a drift. This can be seen as follows.

$$\frac{\partial}{\partial t} \int fg d\mu = (\mathcal{L} f, g) + (f, \mathcal{L} g)$$

$$= ((\mathcal{L} + \mathcal{L}^*) f, g) = 2((Re\mathcal{L}) f, g).$$

Let $L = Re\mathcal{L}$. L has a very natural representation on the cotangent space, using the exterior differentials.

More precisely, let d be the exterior differentiation, defined to be

$$d := \sum_n \partial_{a_n} da_n^{\wedge}.$$

For example, if f is a (scalar) function, then

$$df = \frac{\partial f}{\partial a_1} da_1 + \frac{\partial f}{\partial a_2} da_2 + ... + \frac{\partial f}{\partial a_n} da_n,$$

which can be identified with the vector

$$(\partial f / \partial a_1, \partial f / \partial a_2, ..., \partial f / \partial a_n).$$

A (scalar) function f is called a 0-form; while df a 1-form. There are also higher order forms, e.g., a 2-form:

$$df \wedge dg = -dg \wedge df,$$

which can be identified with an anti-symmetric matrix.

The adjoint d^* is defined to be

$$d^* = \sum_n -\partial_{a_n} da_n^\rfloor.$$

d can be seen as raising the "degree" ; while d^* lowering the degree. (d can be seen as the differentiation operator; while d^* the divergence operator.)

We form the quadratic operator d^*d. It is easy to check that

$$(d^*d)f = -\Delta f,$$

when operating on a scalar function f. So indeed d^*d reduces to the usual Laplacian.

When $\Phi \neq 0$, we define the conjugated d_Φ to be:

$$d_\Phi := e^{-\Phi} d e^{\Phi} = \sum_n (\partial_{a_n} + \partial_{a_n}\Phi) da_n^\wedge;$$

and the adjoint of d_Φ with respect to A to be:

$$d_\Phi^{*,A} = \sum_n A_n(-\partial_{a_n} + \partial_{a_n}\Phi) da_n^\rfloor.$$

Then it is easy to check that the diffusion operator L mentioned earlier can be written in the form:

$$L = d_\Phi^{*,A} d_\Phi,$$

which is a Laplacian on scalar functions, the 0-forms.

What I will say below holds in general, so for illustration purposes, we set $A = Id$.

One of the advantages of this representation is that the Laplacian can be naturally (automatically) generalized to operating on e.g, vector-valued functions, the 1-forms, which occur naturally, and also other higher order forms:

$$\Delta_\Phi = d_\Phi^* d_\Phi + d_\Phi d_\Phi^*.$$

Remark. The second term in the Laplacian is missing in L because $d_\Phi^* f = 0$.

As we will see shortly, within this framework certain integration by parts in deriving the essential inequalities appear "automatic".

We will mostly need Δ_Φ on 0 forms and 1-forms. Writing out explicitly the terms:

$$-\Delta_\Phi^{(1)} = -\Delta_\Phi^{(0)} \otimes \mathbb{I} + 2\Phi''$$

Since the Δ_Φ form a twisted Hodge-complex, there is the intertwining properties of $-\Delta_\Phi^{(0)}$ and $-\Delta_\Phi^{(1)}$, namely

$$\sigma(-\Delta_\Phi^{(0)})\backslash\{0\} \subseteq \sigma(-\Delta_\Phi^{(1)}).$$

This is because if f is an eigenfunction of $-\Delta_\Phi^{(0)}$ with eigenvalue $\lambda \neq 0$, then df is an eigenfunction for $-\Delta_\Phi^{(1)}$ with the **same** eigenvalue λ by using $d_\phi^2 = 0$. (This is sometimes called supersymmetry.)

Using this leads to $L = \Delta_\Phi^{(0)}$ having a spectral gap, provided Φ is convex, which is essentially the spectral (L^2) version of Bacry-Emery.

Remark. The supersymmetry used in random Schrödinger-Poisson statistics, see e.g., **[W, inventiones 2001]**; random walk in a random potential, see e.g., **[W, PTRF 2001]**; nonlinear sigma model/random walk, e.g., **[Disertori-Spencer CMP 2010], [Disertori-Merkl-Rolle, CMP 2014], [Sabot-Tarrés, JEMS 2015]**; and random matrices, see e.g., **[Shcherbyna, 2016]** is related — in the sense that the differential 1-forms provide a representation for the anti-commuting variables, since

$$\xi \wedge \eta = -\eta \wedge \xi.$$

Grassmann integration here is, but integration over differential forms.

The main point is that Grassman integration gives a nice representation for the determinants, which are subsequently more amenable to analysis.

The supersymmetry in Witten-Laplacian, however, has an additional geometric/topological aspect, which so far do not seem to appear in the above mentioned applications.

V. Illustrative proof of the Poincaré (in)equality

Let us consider

$$\int fgd\mu.$$

Without loss, one may assume

$$\langle f\rangle = \langle g\rangle = 0.$$

One may therefore write

$$f = \Delta^{(0)}u,$$

for some u, where for simplicity, we have dropped the sub/superscripts. Then

$$(f,g) = (\Delta^{(0)}u, g)$$

$$
\begin{aligned}
&= (d^* du, g) \\
&= (du, dg) \\
&= (d(\Delta^{(0)})^{-1} f, dg) \\
&= ((\Delta^{(1)})^{-1} df, dg) \\
&= ((\Delta^{(0)} \otimes \mathbb{I} + 2\Phi'')^{-1} df, dg),
\end{aligned}
$$

which maybe called the Poincaré **equality**. If Φ is convex, this leads to the usual Poincaré inequality, after setting $g = f$. $\square$

Remark. The log-Sobolev inequalities for entropy:

$$
\int f \log f - [\int f] \log[\int f],
$$

i.e, the Bacry-Emery (L^1 +log version), can be derived similarly in a slightly lengthier process.