Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2017/2018 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 11 (due Monday 08.01.2018) — Dr. M. Bonacini

Problem 1 (Nemitski operator, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a *Carathéodory function*, that is, f satisfies the following conditions:

a) for every $s \in \mathbb{R}$ the map $x \mapsto f(x,s)$ is measurable in Ω;

b) for almost every $x \in \Omega$ the map $s \mapsto f(x,s)$ is continuous on $\mathbb{R}$.

Let $1 \leq p, q < \infty$, and define the Nemitski composition operator $F : L^p(\Omega) \to L^q(\Omega)$ by setting $F(u)(x) := f(x, u(x))$. Prove that F is well defined and continuous, provided f satisfies

$$|f(x,s)| \leq a(x) + b|s|^{\frac{p}{q}}$$

for some $a \in L^q(\Omega)$ and $b > 0$.

Problem 2 (An application of Schauder's fixed point theorem, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions:

a) for every $s \in \mathbb{R}$ the map $x \mapsto f(x,s)$ is measurable in Ω;

b) for almost every $x \in \Omega$ the map $s \mapsto f(x,s)$ is continuous on $\mathbb{R}$;

c) $|f(x,s)| \leq a(x) + b|s|^\beta$ for every $s \in \mathbb{R}$ and almost every $x \in \Omega$, where $a \in L^2(\Omega)$, $b > 0$, and $0 \leq \beta < 1$.

Use Schauder's fixed point theorem (in particular, the second version proved in Problem 3 in Problem Sheet 10) to show that there exists a weak solution $u \in H_0^1(\Omega)$ to the boundary value problem

$$\begin{cases} -\Delta u = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\beta = 1$, is there always a weak solution to the problem?
Hint: use the properties of Nemitski operators in Problem 1.

Problem 3 (Brezis-Nirenberg example, 2 points).
Let $F : \mathbb{R}^2 \to \mathbb{R}$, $F(x,y) = x^2 + (1-x)^3 y^2$. Show that there are constants $r, a > 0$ such that

$$\inf_{|(x,y)|=r} F(x,y) \geq a \,,$$

and $(x_0, y_0) \in \mathbb{R}^2$ such that

$$|(x_0,y_0)| > r \,, \qquad F(x_0,y_0) \leq 0 \,.$$

What are the critical points of F? Why does the Mountain Pass Theorem fail in this example?

Please turn over.

Problem 4 (Mountain Pass Theorem, 6 points).
Let $\Omega \subset \mathbb{R}^n$ be open, bounded and connected, with smooth boundary, let $k > 0$, and consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda(u-k)^+ & \text{in } \Omega, \\ u \in H_0^1(\Omega), \end{cases} \tag{1}$$

where $(\cdot)^+$ denotes the positive part and $\lambda > \lambda_1$ (first eigenvalue of the Laplacian in Ω). Prove that (1) has a positive solution whose maximum is larger than k, by applying the Mountain Pass theorem to a suitable functional $\Phi : H_0^1(\Omega) \to \mathbb{R}$. In particular, show that:

a) the functional Φ is of class C^1, with Φ' Lipschitz;

b) the functional Φ has the mountain pass geometry;
(Recall that the eigenfunction u_1 corresponding to λ_1 is strictly positive inside Ω.)

c) if $(u_j)_j$ is a Palais-Smale sequence, then $\|u_j\|$ is bounded;
Hint: letting $z_j = u_j/\|u_j\|$ one finds that $z_j \to z$ strongly in $H_0^1(\Omega)$; using also the Maximum Principle, the function z satisfies $z > 0$ and $-\Delta z = \lambda z$, a contradiction.

d) the Palais-Smale condition holds.

Total: 16 points