Nonlinear Partial Differential Equations I | University of Bonn
Winter term 2017/2018 | Prof. Dr. J. J. L. Velázquez
Problem Sheet 8 (due Monday 4.12.2017) | Dr. M. Bonacini

Problem 1 (Hilbert's XIX problem, 3+6+1 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the functional

$$I(u) = \int_\Omega f(Du)\,\mathrm{d}x$$

where $f \in C^\infty(\mathbb{R}^n)$ is uniformly convex, i.e. for some $\lambda > 0$

$$\sum_{i,j=1}^n \frac{\partial^2 f}{\partial p_i \partial p_j}(p)\xi_i\xi_j \geq \lambda|\xi|^2 \qquad \text{for all } p, \xi \in \mathbb{R}^n,$$

and the second derivatives of f are uniformly bounded:

$$\left|\frac{\partial^2 f}{\partial p_i \partial p_j}\right| \leq \Lambda < \infty\,.$$

a) Use the Direct Method of the Calculus of Variations to prove that for given boundary data $u_0 \in H^1(\Omega)$ the functional I has a unique minimizer u in the class

$$X := \{u \in H^1(\Omega) : u - u_0 \in H^1_0(\Omega)\}\,,$$

and that the minimizer u is a weak solution of the Euler-Lagrange equation

$$-\sum_{i=1}^n D_i\Big(\frac{\partial f}{\partial p_i}(Du)\Big) = 0\,. \tag{1}$$

b) Set

$$a_{ij} = \frac{\partial^2 f}{\partial p_i \partial p_j}(Du)\,,$$

and note that the coefficients are bounded and elliptic. Let $1 \leq k \leq n$. Show that the partial derivative $v := \frac{\partial u}{\partial x_k}$ satisfies $v \in H^1_{\mathrm{loc}}(\Omega)$, and is a weak solution to

$$-\sum_{i,j=1}^n D_i\Big(\underbrace{\frac{\partial^2 f}{\partial p_i \partial p_j}(Du(x))}_{a_{ij}(x)}\,D_j v\Big) = 0\,.$$

c) Use the result in part b) to deduce that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, for some $\alpha \in (0,1)$.

Hints for part b): difference quotient method.
Fix any $\varphi \in H^1_0(\Omega)$ with $\operatorname{supp}\varphi \subset\subset \Omega$, and test the weak formulation of equation (1) with the incremental quotient $\tau_{-h,k}\varphi(x) = -\frac{1}{h}(\varphi(x - he_k) - \varphi(x))$, for $|h|$ small. Obtain that the function

$$v_h(x) = \tau_{h,k}u(x) = \frac{u(x + he_k) - u(x)}{h}$$

is a weak solution of a PDE in the form

$$-\sum_{i,j=1}^{n} D_i(b_{ij}^{(h)} D_j v_h) = 0 \tag{2}$$

where the coefficients $b_{ij}^{(h)}$ *are given by*

$$b_{ij}^{(h)} := \int_0^1 f_{p_i p_j}\big(sDu(x+he_k) + (1-s)Du(x)\big)\,\mathrm{d}s\,.$$

Obtain from here a uniform bound $\|Dv_h\|_{L^2(\Omega')} \le C$*, for* $\Omega' \subset\subset \Omega$*. Finally use the general properties of difference quotients (see Evans, Section 5.8.2) to pass to the limit as* $h \to 0$ *in the equation* (2)*.*

Problem 2 (Nonexistence of minimizers: lack of compactness, 3 points).
Let $X = \{u \in W^{1,1}(-1,1) \,:\, u(-1) = -1,\, u(1) = 1\}$, and let

$$F(u) = \int_{-1}^{1} \big(1+|x|\big)|u'(x)|\,\mathrm{d}x\,.$$

Show that $\inf_{u\in X} F(u) = 2$ and that F has no minimizer in X.

Problem 3 (Nonexistence of minimizers: lack of convexity, 3 points).
Let

$$F(u) = \int_0^1 \Big[u(x)^2 + \big(u'(x)^2 - 1\big)^2\Big]\,\mathrm{d}x\,.$$

Show that $\inf_{u\in H_0^1(0,1)} F(u) = 0$ and that F has no minimizer in $H_0^1(0,1)$.

Total: 16 points