Nonlinear Partial Differential Equations I University of Bonn
Winter term 2017/2018 Prof. Dr. J. J. L. Velázquez
Problem Sheet 6 (due Monday 20.11.2017) Dr. M. Bonacini

Problem 1 (Properties of the space BMO, 2+2+2 points).
Let $Q \subset \mathbb{R}^n$ be a cube. The space $BMO(Q)$ is defined as the set of functions $u \in L^1(Q)$ such that

$$[u]_{*,Q} := \sup_{Q' \subset Q} \fint_{Q'} |u - u_{Q'}| \, \mathrm{d}x < \infty \,,$$

where the supremum is over all cubes $Q' \subset Q$ and $u_{Q'}$ denotes the mean value of u on Q'.

a) Prove the inclusion $W^{1,n}(Q) \hookrightarrow BMO(Q)$.

b) Show that $u(x) = \ln x$ belongs to $BMO(Q)$, $Q = (0,1)$.
(Therefore $L^\infty(Q)$ is strictly contained in $BMO(Q)$).

c) Prove that if $u \in BMO(Q)$ then $u \in L^p(Q)$ for every $p \in [1,\infty)$, with

$$\left(\fint_Q |u - u_Q|^p \, \mathrm{d}x \right)^{\frac{1}{p}} \leq c(n,p)[u]_{*,Q} \,.$$

Hint: first prove the equality

$$\|f\|^p_{L^p(Q)} = p \int_0^\infty t^{p-1} |\{x : f(x) > t\}| \, \mathrm{d}t \,.$$

Problem 2 (Harnack inequality in dimension 2, 2+2 points).
We show that, for equations in two variables, the Hölder estimate and the Harnack inequality can be deduced by simpler methods.

a) Let $u \in C^1(B_R)$, $B_R \subset \mathbb{R}^2$, and write for $0 < r < R$

$$\omega(r) := \operatorname*{osc}_{\partial B_r} u = \sup_{\partial B_r} u - \inf_{\partial B_r} u \,.$$

If ω is non-decreasing, show that

$$\omega(r) \leq \sqrt{\frac{\pi}{\ln(R/r)}} \left(\int_{B_R} |Du|^2 \, \mathrm{d}x \right)^{\frac{1}{2}} .$$

b) Let $u \in C^1(B_R)$ be a nonnegative solution to the elliptic equation

$$-\sum_{i,j=1}^{2} D_i(a_{ij} D_j u) = 0 \,, \qquad \text{with } a_{ij} \in L^\infty(B_R), \quad \sum_{i,j=1}^{2} a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2 \,.$$

Prove the Harnack inequality

$$\sup_{B_r} u \leq C \inf_{B_r} u \,,$$

for $0 < r < R$, where C is a constant depending on the coefficients a_{ij}, on r, and on R.
Hint: first show that the Dirichlet integral $\int_{B_r} |Dv|^2 \, \mathrm{d}x$ *of the function* $v = \ln u$ *is bounded in every disc* B_r, $0 < r < R$. *Then apply part a) to obtain the result.*

Problem 3 (Decay estimates for constant coefficients equations, 6 points).
Let $\Omega \subset \mathbb{R}^n$ be open and let $u \in H^1_{\mathrm{loc}}(\Omega)$ be a weak solution to the equation

$$-\sum_{i,j=1}^{n} D_i(a_{ij} D_j u) = 0 \qquad \text{locally in } \Omega\,,$$

where the coefficients a_{ij} are constant and satisfy the estimates

$$|a_{ij}| \le \Lambda\,, \qquad\qquad \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \ge \lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n\,.$$

Prove that, for all $x_0 \in \Omega$ and $0 < r < R < \mathrm{dist}\,(x_0, \partial\Omega)$, u satisfies

$$\int_{B_r(x_0)} |u|^2 \,\mathrm{d}x \le c\Big(\frac{r}{R}\Big)^n \int_{B_R(x_0)} |u|^2 \,\mathrm{d}x\,, \tag{1}$$

$$\int_{B_r(x_0)} |u - u_{x_0,r}|^2 \,\mathrm{d}x \le c\Big(\frac{r}{R}\Big)^{n+2} \int_{B_R(x_0)} |u - u_{x_0,R}|^2 \,\mathrm{d}x\,, \tag{2}$$

where $u_{x_0,\rho} = \frac{1}{|B_\rho|}\int_{B_\rho} u \,\mathrm{d}x$, and c is a constant depending only on n, λ and Λ.

Hints: to prove (1)*, argue as follows:*

a) Consider first the case $r < \frac{R}{2}$. Use the Hilbert-space Regularity Theory, together with the embedding $H^k(B_\rho) \hookrightarrow L^\infty(B_\rho)$ for k large, to prove that

$$\frac{1}{r^n}\int_{B_r(x_0)} |u|^2 \,\mathrm{d}x \le c(n, \lambda, \Lambda, R) \int_{B_R(x_0)} |u|^2\,.$$

b) Use a scaling argument to make explicit the dependence of c on R: assume without loss of generality that $x_0 = 0$, and observe that for a given solution u in B_R, the rescaled function $\tilde{u}(x) = u(Rx)$ is a solution in B_1...

c) Finally, consider the case $r \ge \frac{R}{2}$.

To prove (2)*: apply the first estimate* (1) *to Du (why can you do this?). Then combine this inequality with Caccioppoli inequality (see Exercise 3a in Problem Sheet 4).*

Total: 16 points

The student council of mathematics will organize the math party on 23/11 in N8schicht. The presale will be held on Mon 20/11, Tue 21/11 and Wed 22/11 in the mensa Poppelsdorf. Further information can be found at fsmath.uni-bonn.de.