
Problem 1 (Condition for Hölder continuity, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^\infty_{\mathrm{loc}}(\Omega)$. For $x \in \Omega$ and $r < \operatorname{dist}(x, \partial\Omega)$ define the *oscillation* of u in the ball $B_r(x)$ as

$$\omega(x,r) := \operatorname*{ess\,sup}_{B_r(x)} u - \operatorname*{ess\,inf}_{B_r(x)} u .$$

Prove that, if there exists $\sigma \in (0,1)$ such that for every $x \in \Omega$ and $r \in (0, \frac{1}{4}\operatorname{dist}(x, \partial\Omega))$

$$\omega(x,r) \le (1-\sigma)\omega(x,4r),$$

then there exists $\bar{u} \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, for $\alpha = \log_4 \frac{1}{1-\sigma}$, such that $u = \bar{u}$ almost everywhere in Ω (that is, u has a locally Hölder continuous representative).

Problem 2 (Limit of L^p-norm, 2 points).
Let u be a measurable function in a bounded open set $\Omega \subset \mathbb{R}^n$, $u > 0$. Prove that

$$\operatorname*{ess\,sup}_{x\in\Omega} u(x) = \lim_{p\to\infty} \left(\int_\Omega u^p \,\mathrm{d}x \right)^{\frac{1}{p}}, \qquad \operatorname*{ess\,inf}_{x\in\Omega} u(x) = \lim_{p\to-\infty} \left(\int_\Omega u^p \,\mathrm{d}x \right)^{\frac{1}{p}}.$$

Problem 3 (Maximum principle, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and consider the elliptic operator

$$Lu = -\sum_{i,j=1}^n a_{ij} D_{ij} u + \sum_{j=1}^n b_j D_j u + cu ,$$

where $a_{ij}, b_j, c \in C(\overline{\Omega})$, $a_{ij} = a_{ji}$,

$$\sum_{i,j=1}^n a_{ij}\xi_i\xi_j \ge \alpha|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n , \qquad c \ge 0 .$$

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy the inequality $Lu \le 0$ in Ω.

a) Assume $c = 0$ and show that

$$\max_{\overline{\Omega}} u = \max_{\partial\Omega} u .$$

Hint: consider first the case $Lu < 0$; then consider $u + \varepsilon e^{\gamma x_1}$ for suitable $\gamma > 0$.

b) If $c \ge 0$, show that

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u^+ .$$

Please turn over.

Problem 4 (Weak formulation of the maximum principle, 6 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $u \in H^1(\Omega)$ set $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$.

a) Let $Lu = -\sum_{i,j=1}^n D_i\big(a_{ij} D_j u\big)$, where the coefficients $a_{ij} \in L^\infty(\Omega)$ are elliptic. Assume that $u \in H^1(\Omega)$ is a (weak) subsolution for the operator L, that is $Lu \leq 0$ in the sense

$$\sum_{i,j=1}^n \int_\Omega a_{ij} D_j u D_i \varphi \, \mathrm{d}x \leq 0 \qquad \text{for every } \varphi \in H_0^1(\Omega) \text{ with } \varphi \geq 0.$$

Prove that, if $(u - M)^+ \in H_0^1(\Omega)$, then $u \leq M$ almost everywhere in Ω ($M \in \mathbb{R}$).

In particular we have the comparison principle: if $u, v \in H^1(\Omega)$ satisfy $Lu \leq 0$, $Lv \geq 0$, and $(u - v)^+ \in H_0^1(\Omega)$, then $u \leq v$ almost everywhere in Ω.

b) Define
$$H_-^1(\Omega) = \left\{ u \in H^1(\Omega) : u^+ \in H_0^1(\Omega) \right\}.$$

Show that $u \in H_-^1(\Omega)$ if and only if there exist $u_m \in H^1(\Omega)$ and compact sets $K_m \subset \Omega$ such that
$$u_m \leq 0 \quad \text{in } \Omega \backslash K_m\,, \qquad\qquad u_m \to u \quad \text{in } H^1(\Omega).$$

Hint: note that $u = u^+ - u^-$. You can use without proof that $f_m \to f$ in $H^1(\Omega)$ implies $f_m^+ \to f^+$ in $H^1(\Omega)$.

c) Show that, if $u \in H^1(\Omega) \cap C(\overline{\Omega})$ and $u \leq 0$ on $\partial\Omega$, then $u \in H_-^1(\Omega)$.

(Therefore this means that the condition $(u - M)^+ \in H_0^1(\Omega)$ is a "weak form" of writing that $u \leq M$ on $\partial\Omega$, when we don't have sufficient regularity of u and $\partial\Omega$).

Total: 16 points