Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2017/2018 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 4 (due Monday 6.11.2017) — Dr. M. Bonacini

Problem 1 (Eigenvalues of symmetric, elliptic operators, 2+2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider an elliptic operator

$$Lu = -\sum_{i,j=1}^{n} D_i(a_{ij}D_ju)\,,$$

where the coefficients $a_{ij} \in L^\infty(\Omega)$ are symmetric and uniformly elliptic, and the associated symmetric bilinear form

$$B(u,v) = \sum_{i,j=1}^{n}\int_\Omega a_{ij}D_juD_iv\,\mathrm{d}x, \qquad u,v \in H_0^1(\Omega).$$

a) Prove that the operator L (with Dirichlet boundary conditions) has a sequence of real, positive eigenvalues

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \dots, \qquad\qquad \lambda_k \to \infty \quad \text{as } k \to \infty,$$

Moreover there is a countable orthonormal basis $\{u_k\}_{k\ge1}$ of $L^2(\Omega)$, where $u_k \in H_0^1(\Omega)$ is an eigenfunction corresponding to λ_k:

$$\begin{cases} Lu_k = \lambda_k u_k & \text{in } \Omega,\\ u_k = 0 & \text{on } \partial\Omega. \end{cases}$$

(Recall the spectral theory for symmetric compact operators.)

b) Prove the formula

$$\lambda_k = \min\left\{ \frac{B(u,u)}{\|u\|_{L^2}^2} \;:\; u \in H_0^1(\Omega),\ (u,u_i)_{L^2} = 0 \text{ for } i = 1,\dots,k-1 \right\},$$

where $(u,v)_{L^2}$ denotes the scalar product in $L^2(\Omega)$.
Hint: first show that the functions $w_k := \frac{1}{\sqrt{\lambda_k}}u_k$ *form an orthonormal basis of* $H_0^1(\Omega)$ *with respect to the scalar product* $B(\cdot,\cdot)$.

c) (*Courant minmax principle*) Prove that

$$\lambda_k = \sup_{\varphi_1,\dots,\varphi_{k-1}\in L^2(\Omega)} \min\left\{ \frac{B(u,u)}{\|u\|_{L^2}^2} \;:\; u \in H_0^1(\Omega),\ (u,\varphi_i)_{L^2} = 0 \text{ for } i = 1,\dots,k-1 \right\}.$$

Hint: choose $u = \sum_{i=1}^{k}\beta_iu_i$ *for suitable* β_i.

Please turn over.

Problem 2 (Monotone dependence of eigenvalues on the domain, 4 points).
Let $a_{ij} \in L^\infty(\mathbb{R}^n)$ be symmetric and uniformly elliptic coefficients, and consider the elliptic operator $Lu := -\sum_{i,j=1}^n (a_{ij}u_{x_j})_{x_i}$. For a given open and bounded set $\Omega \subset \mathbb{R}^n$, denote by $\{\lambda_k(\Omega)\}_k$ the sequence of eigenvalues of L (with Dirichlet boundary conditions) on the domain Ω: that is, the values for which there exist nontrivial solutions to the boundary value problem

$$\begin{cases} Lu = \lambda_k(\Omega)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Show that if $\Omega_1 \subset \Omega_2$ are open and bounded, then $\lambda_k(\Omega_2) \leq \lambda_k(\Omega_1)$ for every k.
Hint: use the two characterizations of λ_k given in Problem 1.

Problem 3 (Widman's hole-filling technique, 2+2+1+1 points).
Let $u \in H^1_{\text{loc}}(\Omega)$ be a weak solution to the elliptic equation $-\sum_{i,j=1}^n D_i(a_{ij}D_j u) = 0$ in an open set $\Omega \subset \mathbb{R}^n$, where the coefficients $a_{ij} \in L^\infty(\Omega)$ satisfy for constants $0 < \lambda < \Lambda < \infty$

$$\sup_x |a_{ij}(x)| \leq \Lambda\,, \qquad \sum_{i,j=1}^n a_{ij}\xi_i\xi_j \geq \lambda|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.$$

a) Prove the Caccioppoli inequality: for all $x \in \Omega$, $0 < r < R < \operatorname{dist}(x, \partial\Omega)$ and $t \in \mathbb{R}$

$$\int_{B_r(x)} |Du|^2 \leq 16\left(\frac{\Lambda}{\lambda}\right)^2 \frac{1}{(R-r)^2} \int_{B_R(x)\setminus B_r(x)} |u-t|^2\,.$$

b) Deduce that there exists $\theta < 1$ such that for all $x \in \Omega$ and $0 < R < \operatorname{dist}(x, \partial\Omega)$

$$\int_{B_{R/2}(x)} |Du|^2 \leq \theta \int_{B_R(x)} |Du|^2\,.$$

c) Show that for all $x \in \Omega$ and $0 < r < R < \operatorname{dist}(x, \partial\Omega)$

$$\int_{B_r(x)} |Du|^2 \leq 2^\alpha \left(\frac{r}{R}\right)^\alpha \int_{B_R(x)} |Du|^2$$

with $\alpha = \log_2(1/\theta)$.
Remark: for $n = 2$, this implies that $u \in C^{0,\alpha/2}$.

d) Prove Liouville's Theorem: if $u \in H^1_{\text{loc}}(\mathbb{R}^2)$ is a bounded weak solution to the elliptic equation $-\sum_{i,j=1}^n D_i(a_{ij}D_j u) = 0$ in $\mathbb{R}^2$, then u is constant.

Please turn over.

Problem 4 (BONUS: extra credit 3 points).
The following property of Sobolev Spaces has been used in the lecture:
Let $\Omega \subset \mathbb{R}^n$ be an open, bounded set and let $u \in W^{1,p}(\Omega)$, $1 \leq p \leq \infty$.

a) Show that $u^+, u^-, |u| \in W^{1,p}(\Omega)$, where $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$, and that

$$Du^+(x) = \begin{cases} Du(x) & \text{if } u(x) > 0, \\ 0 & \text{if } u(x) \leq 0, \end{cases}$$

$$Du^-(x) = \begin{cases} 0 & \text{if } u(x) \geq 0, \\ Du(x) & \text{if } u(x) < 0. \end{cases}$$

b) Show that $Du = 0$ almost everywhere on the set $\{x \in \Omega : u(x) = c\}$, where $c \in \mathbb{R}$ is any given constant.

Total: 16 points, extra credit 3 points