Nonlinear Partial Differential Equations I University of Bonn
Winter term 2017/2018 Prof. Dr. J. J. L. Velázquez
Problem Sheet 3 (due Monday 30.10.2017) Dr. M. Bonacini

Problem 1 (Regularity for a semilinear problem, 2+2+2 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded. Consider the elliptic operator $Lu = -\sum_{i,j=1}^{n}(a_{ij}u_{x_j})_{x_i}$, where the coefficients $a_{ij} \in C^2(\Omega)$ are symmetric and uniformly elliptic.

a) Suppose that $f \in C^1(\mathbb{R})$ satisfies $\|f'\|_\infty < \infty$. Assume that $u \in H^1(\Omega)$ is a weak solution to
$$Lu = f(u) \quad \text{in } \Omega. \tag{1}$$
Show that $u \in H^3_{\mathrm{loc}}(\Omega)$.

b) Assume further that $a_{ij} \in C^3(\Omega)$ and $f \in C^2(\mathbb{R})$ with $\|f''\|_\infty < \infty$. Prove that $u \in H^4_{\mathrm{loc}}(\Omega)$, provided that the dimension n of the space is not too large.
Hint: Sobolev Embeddings.

c) Let $f(u) = |u|^p$, $p \geq 1$. For which values of p can you write a weak formulation of the equation (1) in $H^1_0(\Omega)$? For which values of p can you ensure that a weak solution u to (1) belongs to $H^2_{\mathrm{loc}}(\Omega)$?

Problem 2 (Boundary regularity, 4+2 points).
In $\mathbb{R}^2$ use the polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$ and define the angular domain
$$\Omega := \left\{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \,:\, r \in (0,1),\, \theta \in (0,\omega)\right\},$$
for $\omega \in (0, 2\pi)$.

a) Check that the function $u(x,y) = r^{\frac{\pi}{\omega}} \sin(\frac{\pi}{\omega}\theta)$ lies in $H^1(\Omega)$ and solves
$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial_D\Omega := \{(r\cos\theta, r\sin\theta) \,:\, 0 \leq r \leq 1,\, \theta \in \{0,\omega\}\}, \\ \frac{\partial u}{\partial \nu} = \frac{\pi}{\omega}\sin(\frac{\pi}{\omega}\theta) & \text{on } \partial_N\Omega := \partial\Omega \setminus \partial_D\Omega \end{cases}$$
(where $\frac{\partial u}{\partial \nu} = Du \cdot \nu$ denotes the derivative of u in the normal direction ν to $\partial_N\Omega$). For which values of ω do we have $u \in H^2(\Omega)$? If $u \notin H^2(\Omega)$ what is the problem?

b) Find a function $f \in C^0(\overline{\Omega})$ such that the unique solution w of the Dirichlet problem $\Delta w = f$ in Ω and $w = 0$ on $\partial\Omega$ lies in $H^1(\Omega)$ but not in $H^2(\Omega)$.
Hint: write $w = u - v$ for a smooth function v.

Please turn over.

Problem 3 (Convergence of weak solutions, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\Omega_k \subset \Omega$ be a sequence of open subsets of Ω such that $\overline{\Omega}_k \subset \Omega_{k+1}$ and $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Let $w \in H^1(\Omega)$ and let $u_k \in H^1(\Omega_k)$ be the unique weak solution to

$$\begin{cases} \Delta u_k = 0 & \text{in } \Omega_k, \\ u_k - w \in H_0^1(\Omega_k). \end{cases}$$

By setting

$$\tilde{u}_k(x) := \begin{cases} u_k & \text{in } \Omega_k, \\ w & \text{in } \Omega \backslash \Omega_k, \end{cases}$$

show that $\tilde{u}_k$ converges strongly in $H^1(\Omega)$ to the unique weak solution u to

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u - w \in H_0^1(\Omega). \end{cases}$$

Total: 16 points