Nonlinear Partial Differential Equations I | University of Bonn
Winter term 2017/2018 | Prof. Dr. J. J. L. Velázquez
Problem Sheet 2 (due Monday 23.10.2017) | Dr. M. Bonacini

Problem 1 (Stampacchia's Theorem, 6 points).
The goal of the exercise is to prove the following generalization of Lax-Milgram Theorem:

Let H be a Hilbert space. Let $a : H \times H \to \mathbb{R}$ be a continuous and linear form <u>in the second variable</u> such that for every $u_1, u_2, v \in H$

(i) $|a(u_1, v) - a(u_2, v)| \leq \beta \|u_1 - u_2\| \|v\|$,

(ii) $a(u_1, u_1 - u_2) - a(u_2, u_1 - u_2) \geq \gamma \|u_1 - u_2\|^2$,

where β and γ are positive constants. Then for every $F : H \to \mathbb{R}$ linear and continuous there exists a unique $u \in H$ such that $a(u, v) = F(v)$ for every $v \in H$.

To prove the theorem, proceed in two steps:

1) Show that if $A : H \to H$ satisfies the two conditions

   (a) $\|A(u_1) - A(u_2)\| \leq \beta \|u_1 - u_2\|$

   (b) $\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq \gamma \|u_1 - u_2\|^2$

   for all $u_1, u_2 \in H$, then for every $f \in H$ there is a unique $u_f \in H$ such that $A(u_f) = f$ (here $\langle \cdot, \cdot \rangle$ denotes the scalar product in H).
   Hint: apply Banach's Fixed Point Theorem to the map $R(u) = u - \lambda A(u) + \lambda f$, for suitable $\lambda > 0$.

2) Use Riesz's Representation Theorem to define a suitable map $A : H \to H$ and apply the previous step.

Problem 2 (Semilinear monotone equations, 2 points).
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, let $a_{ij} \in L^\infty(\Omega)$ be uniformly elliptic coefficients, and let $f \in L^2(\Omega)$. Let $g : \mathbb{R} \to \mathbb{R}$ be increasing and Lipschitz continuous, that is

$$|g(t) - g(s)| \leq L|t - s| \qquad \text{for all } s, t \in \mathbb{R}.$$

Use Stampacchia's Theorem (Problem 1) to show that there exists a unique weak solution $u \in H^1_0(\Omega)$ to the semilinear boundary value problem

$$\begin{cases} -\sum_{i,j=1}^n D_i\big(a_{ij} D_j u\big) + g(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Please turn over.

Problem 3 (Weak solutions as minimizers, 4 points).
Let H be an Hilbert space, let $B : H \times H \to \mathbb{R}$ be a symmetric, bilinear form such that $B(u,u) \geq 0$ for every $u \in H$, and let $F : H \to R$ be a linear functional. Prove that $u \in H$ satisfies

$$B(u,v) = F(v) \qquad \text{for every } v \in H$$

if and only if u is minimizer of the problem

$$\min\left\{ \frac{1}{2}B(v,v) - F(v) \,:\, v \in H \right\}.$$

Apply the previous property to show that every weak solution $u \in H_0^1(\Omega)$ to the elliptic equation

$$-\sum_{i,j=1}^{n} D_i\big(a_{ij} D_j u\big) = f$$

(with $a_{ij} \in L^\infty(\Omega)$ symmetric and elliptic, $f \in L^2(\Omega)$) is a minimizer of a suitable functional.

Problem 4 (Neumann boundary conditions, 4 points).
Let $\Omega \subset \mathbb{R}^n$ be an open, bounded, connected set with C^1 boundary, and consider the operator

$$Lu(x) := -\sum_{i,j=1}^{n} D_i\big(a_{ij}(x) D_j u(x)\big)$$

where the coefficients $a_{ij} \in L^\infty(\Omega)$ are uniformly elliptic.

(i) Show that, for every $f \in L^2(\Omega)$ with $\int_\Omega f(x)\,\mathrm{d}x = 0$, there exists a weak solution $u \in H^1(\Omega)$ of the problem $Lu = f$, in the sense that

$$\sum_{i,j=1}^{n} \int_\Omega a_{ij} D_j u D_i v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x \qquad \text{for every } v \in H^1(\Omega),$$

and that such solution is unique up to constants (that is, the difference of any two weak solutions is a constant).

(ii) Assume that $a_{ij}, u \in C^\infty(\overline{\Omega})$. Which boundary condition does u satisfy? What is this condition in the case of the Laplace operator $Lu = -\Delta u$?

Total: 16 points