Nonlinear Partial Differential Equations II University of Bonn
Summer term 2016 Prof. Dr. J.J.L. Velázquez
Problem Sheet 11 (due Monday 04.07.2016)

Problem 1 (Self similarity, 4 points).
Find the parameters α, β such that the differential equations

$$u_t = u_{xx} + (u^2)_x$$

admits a similarity solution in the form $u(x,t) = t^\alpha f(\frac{x}{t^\beta})$. Find a differential equation for f and show that this ODE can be reduced to first order.

Problem 2 (A nonlinear diffusion equation, 8 points).
Consider the nonlinear diffusion equation

$$u_t - (uu_x)_x = 0 \qquad (x,t) \in \mathbb{R} \times (0,\infty),$$

subject to the initial condition of a unit point source applied at $x = 0$ at time $t = 0$, that is, $u(x,0) = \delta(x)$ (where δ is the Dirac delta), with

$$\int_{\mathbb{R}} u(x,t)\,\mathrm{d}x = 1\,, \qquad \lim_{x\to\pm\infty} u(x,t) = 0 \tag{1}$$

for all $t > 0$. We look for a self similar solution

$$u(x,t) = t^\alpha w(xt^\beta)\,.$$

a) Determine the values of the parameters α, β and the equation solved by w.
Hint: impose also the first condition in (1).

b) Show that the equation for w can be integrated once, to get the ODE

$$3w(y)w'(y) + yw(y) = 0\,.$$

Obtain a piecewise smooth solution satisfying also (1).
Hint: you can use the fact that, because the solution must be symmetric around 0, $u_x(0,t) = 0$ *and hence* $w'(0) = 0$.

c) Go back to the original variables and write down a piecewise solution $u(x,t)$. Sketch the function $u(\cdot,t)$ for a few values of t.

Problem 3 (Self-similarity in Burger's equation, 4 points).
Consider Burger's equation

$$u_t + uu_x = 0 \qquad (x,t) \in \mathbb{R} \times (0,\infty)$$

with $u(x,0) = \delta(x)$, where δ is the Dirac delta. Recall that by the conservation property

$$\int_{-\infty}^{+\infty} u(x,t)\ \mathrm{d}x = 1 \qquad \text{for all } t \geq 0.$$

Find a (piecewise smooth) self-similar solution. In particular, check that the Rankine-Hugoniot condition is satisfied by your solution.
Hint: the solution should be a N-wave.

Total: 16 points