

Problem 1 (N-wave, 4 points).

Let u be the entropy solution to

$$\begin{cases} u_t + \partial_x \left(\frac{|u|^p}{p} \right) = 0 & \text{in } \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x), \end{cases}$$

where $p \in (2, \infty)$ and

$$u_0(x) = \begin{cases} -1 & \text{for } -1 \leq x < 0, \\ 1 & \text{for } 0 \leq x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the asymptotic shape $u(\cdot, t)$ as $t \rightarrow \infty$.

Problem 2 (Vanishing viscosity, 8 points).

Let F be smooth and uniformly convex, $g \in L^\infty(\mathbb{R})$ be compactly supported, $T > 0$. Let $\varepsilon > 0$, set $M = \varepsilon^{-1}$, and suppose that $u^\varepsilon \in C^\infty([-M, M] \times [0, T])$ is a solution to the viscous conservation law

$$\begin{cases} u_t^\varepsilon + F(u^\varepsilon)_x = \varepsilon u_{xx}^\varepsilon & \text{for } (x, t) \in (-M, M) \times (0, T], \\ u_x^\varepsilon = 0 & \text{for } (x, t) \in \{\pm M\} \times (0, T], \\ u^\varepsilon = g^\varepsilon & \text{for } (x, t) \in [-M, M] \times \{0\}, \end{cases} \quad (1)$$

where $g^\varepsilon = \eta^\varepsilon * g$ and η^ε is the standard mollifier.

a) Show that

$$\|u^\varepsilon\|_{L^\infty([-M, M] \times [0, T])} \leq \|g\|_{L^\infty(\mathbb{R})}. \quad (2)$$

Hint: use a maximum principle argument.

b) Since F is uniformly convex, there exists $\theta > 0$ such that $F'' \geq \theta$. Show that

$$u_x^\varepsilon(x, t) \leq \frac{1}{\theta t} \quad \text{for all } (x, t) \in [-M, M] \times (0, T]. \quad (3)$$

Hint: set $a(x, t) := u_x^\varepsilon(x, t)$ and find a PDE solved by a by differentiating (1). Show that $v(x, t) := \frac{1}{\theta t}$ is a supersolution of this PDE and conclude by a comparison principle argument (similar to the one used in Problem 2 in the Problem Sheet 5).

c) Assume that

$$\lim_{\varepsilon \rightarrow 0^+} u^\varepsilon(x, t) = u(x, t) \quad \text{for almost every } (x, t) \in \mathbb{R} \times [0, T].$$

Show that the limit function u is the entropy solution to the conservation law

$$\begin{cases} u_t + F(u)_x = 0 & \text{for } (x, t) \in \mathbb{R} \times (0, T), \\ u = g & \text{for } (x, t) \in \mathbb{R} \times \{0\}. \end{cases}$$

Hint: check that u is an integral solution by multiplying (1) by a test function and passing to the limit as $\varepsilon \rightarrow 0$. Moreover, use (3) to show that u satisfies the entropy condition.

Problem 3 (A linear system, 4 points).

Consider the system of conservation laws

$$\partial_t u + \partial_x (Au) = 0, \quad (x, t) \in \mathbb{R} \times (0, \infty), \quad (4)$$

where $u : \mathbb{R} \times [0, \infty) \rightarrow \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$ is a diagonalizable matrix with real eigenvalues.

- a) Find the Rankine-Hugoniot condition along a shock curve. What does it mean?
- b) By a linear transformation, show that (4) decouples into a system of n independent equations.

Total: 16 points