Nonlinear Partial Differential Equations II

Summer term 2016

Problem Sheet 3 (due Monday 02.05.2016)

University of Bonn Prof. Dr. J.J.L. Velázquez

Problem 1 (Method of characteristics, 2 points).

Use the method of characteristics to solve the nonlinear initial value problem

$$\begin{cases} u_t + (u_x)^4 = 0 & (x,t) \in \mathbb{R} \times (0,\infty), \\ u(x,0) = \frac{3}{4}x^{\frac{4}{3}}. \end{cases}$$

Problem 2 (A nonlinear PDE, 4 points).

Consider the initial value problem

$$\begin{cases} -(u_{x_1})^2 + (u_{x_2})^2 + x_2^2 = 0 & \text{in } \mathbb{R}^2, \\ u(x_1, 0) = g(x_1), & \end{cases}$$

where q is a C^1 -function satisfying $g'(x_1) > 0$ for every x_1 .

- a) Find explicitly the projected characteristics $(x_1(s), x_2(s))$ starting from the point (y, 0), $y \in \mathbb{R}$, assuming $u_{x_2}(x, 0) \geq 0$.
- b) In the case $g(x_1) = x_1$, sketch the projected characteristics and write down an explicit solution in a neighborhood of the x_1 -axis (your answer may depend on the function $I(s) = \int_0^s \sin^2(s) \, ds$).

Problem 3 (Blow up in Burgers' equation, 3+3 points).

Let $u_0 \in C^{\infty}(\mathbb{R})$ be such that u'_0 is bounded and has a unique minimizer s_0 with $u'_0(s_0) < 0$, $u''_0(s_0) = 0$, $u'''_0(s_0) > 0$.

a) Show that there is $0 < t^* < \infty$ such that the solution to the initial value problem

$$\begin{cases} \partial_t u + u \partial_x u = 0 & \text{for } t \in (0, t^*), x \in \mathbb{R}, \\ u(0, x) = u_0(x) \end{cases}$$
 (1)

is given by

$$u(t,x) = u_0(\xi), \qquad x = \xi + u_0(\xi)t.$$

Determine the largest t^* , which depends on some $\xi^* \in \mathbb{R}$, such that u as a function of (t,x) exists and solves (1). What happens with $\partial_t u, \partial_x u$ and the characteristic curves of (1) as $t \to t^*$?

b) Perform a local analysis (Taylor expansion, term comparison) to argue that

$$u(t^*, x) - u_0(\xi^*) \sim -(x - x^*)^{\frac{1}{3}}$$

for x close to x^* , where $x^* = \xi^* + u_0(\xi^*)t^*$. Here $a \sim b$ means a = Cb for some constant C > 0, up to higher order terms.

Problem 4 (Eikonal equation on a strip, 4 points).

Consider the Eikonal equation

$$|Du| = ((\partial_{x_1} u)^2 + (\partial_{x_2} u)^2)^{\frac{1}{2}} = 1$$

in the strip $U = \{0 < x_1 < 1\} \subset \mathbb{R}^2$, with boundary data $u(0, x_2) = u(1, x_2) = 0$ for $x_2 \in \mathbb{R}$.

- a) Write the characteristic ODE and determine admissible initial data. (Note that at each point of the boundary you have two possible sets of initial data).
- b) Choose the initial data $p_1(0) = 1$ on the left boundary $\{x_1 = 0\}$, and $p_1(0) = -1$ on the right boundary $\{x_1 = 1\}$. Explicitly solve the characteristic ODE.
- c) Let $(x_1(s,x_0),x_2(s,x_0))$ be the projected characteristics starting from a boundary point $x_0 \in \partial U$, found in b). Notice that, given $(x_1,x_2) \in U$, you cannot uniquely determine $x_0 \in \partial U$ and s such that $(x_1,x_2) = (x_1(s,x_0),x_2(s,x_0))$. What solution do you construct if you assign to (x_1,x_2) the characteristic starting from the left boundary if $x_1 < \alpha$, and the one starting from the right boundary if $x_1 > \alpha$, where $\alpha \in (0,1)$ is fixed? Is one solution "better" than the others?

Total: 16 points