Nonlinear Partial Differential Equations II University of Bonn
Summer term 2016 Prof. Dr. J.J.L. Velázquez
Problem Sheet 2 (due Monday 25.04.2016)

Problem 1 (Method of characteristics, 2+2 points).
Use the method of characteristics to solve the following equations:

a) $$\begin{cases} x_1\partial_{x_1}u - x_2\partial_{x_2}u + u = x_1 & \text{in } \mathbb{R}^2, \\ u(x_1,x_2) = x_1 & \text{on } \Gamma = \{x_2 = {x_1}^2\}. \end{cases}$$

b) $$\begin{cases} u\partial_{x_1}u + \partial_{x_2}u = 1 & \text{in } \mathbb{R}^2, \\ u(x_1,x_2) = \frac{1}{2}x_1 & \text{on } \Gamma = \{x_2 = x_1\}. \end{cases}$$

Problem 2 (A quasilinear PDE, 4 points).
Show that the solution to the quasilinear PDE

$$u_y + a(u)u_x = 0$$

with initial condition $u(x,0) = h(x)$ is given implicitly by

$$u(x,y) = h(x - a(u)y).$$

Show that the solution becomes singular for some positive y, unless $a(h(s))$ is a nondecreasing function of s.

Problem 3 (Linear transport, 4 points).
Consider the linear transport equation

$$\partial_t u(t,x) + a(t,x)\partial_x u(t,x) = 0.$$

a) Determine the characteristic curves $x = x(t)$, $x(0) = x_0$ for the transport velocities

   (1) $a(t,x) = x$,
   (2) $a(t,x) = \omega\cos(\omega t + \phi)$, $\omega, \phi \in \mathbb{R}$,
   (3) $a(t,x) = g(x_0)$, where
   $$g(x_0) = \begin{cases} 0 & \text{if } x_0 \le 0, \\ -x_0 & \text{if } 0 < x_0 < 1, \\ -1 & \text{if } x_0 \ge 1. \end{cases}$$

   In each case, sketch the trajectories $x(t)$ for $t > 0$ and several values of x_0.

b) Give a simple yet nontrivial condition on $a(t,x)$ such that

   (1) characteristic curves do not intersect,

(2) a solution exists for all $t > 0$.

Problem 4 (4 points).
Suppose $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ is a smooth solution of $u_t + uu_x = 0$ which is periodic in x with period $L > 0$, i.e., $u(x + L, t) = u(x, t)$. Show that

$$\max_x u(x, 0) - \min_x u(x, 0) \leq L/T\,.$$

Total: 16 points