Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 12 (due Thursday 28.01.16 if you want it marked)

Problem 1
Let $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for $1 \le p < \infty$. Show that

$$\lim_{\substack{B \supset \{x\} \\ \mathrm{diam} B \to 0}} \fint_B |f(y) - f(x)|^p \,\mathrm{d}y = 0$$

for almost every $x \in \mathbb{R}^n$, where the limit is taken over balls that contain x as the diameter tends to 0.
Hint: First, use the Lebesgue-differentiation theorem from the lecture to prove the claim for cubes $Q(x,r)$; to this end, consider

$$\lim_{r \to 0} \fint_{Q(x,r)} |f(y) - q_i|^p \,\mathrm{d}y$$

for suitable $\{q_i\} \subset \mathbb{R}$. Then, extend the result to balls not necessarily centred at x.

Problem 2
Let $A \subset \mathbb{R}^n$ be Lebesgue measurable.

1. Show that
$$\lim_{r \to 0} \frac{|B_r(x) \cap A|}{|B_r(x)|} = 1 \quad \text{for a. e. } x \in A, \quad \lim_{r \to 0} \frac{|B_r(x) \cap A|}{|B_r(x)|} = 0 \quad \text{for a. e. } x \in \mathbb{R}^n \setminus A.$$

2. Assume in addition that $|A| > 0$. Show that the difference set
$$\{x - y : x, y \in A\}$$
contains a ball centred at the origin.

Problem 3 (Mathematical billiard)
For the purpose of this problem, billiard is a nonempty, bounded, strictly convex set $\Omega \subset \mathbb{R}^2$ with smooth boundary such that a frictionless ball is bouncing around in it. The orbits are sequences of line segments where each two successive segments share a point on the boundary $\partial\Omega$, and at this point the two segments make the same angle with the tangent to the boundary (reflection).
Let $\gamma \colon I \to \mathbb{R}^2$, $I \subset \mathbb{R}^2$ be a smooth parametrization of $\partial\Omega$ and denote by

$$h(s, s') = |\gamma(s) - \gamma(s')|, \qquad s, s' \in I$$

the distance in $\mathbb{R}^2$ between the boundary points $\gamma(s)$ and $\gamma(s')$.

1. An n-periodic orbit is a closed orbit with n boundary points. Show that the variational formulation of finding n-periodic orbits is finding critical points of
$$H_n(s_1, \ldots, s_n) = \sum_{k=1}^{n} h(s_k, s_{k+1})$$
where $s_{n+1} = s_n$.

2. Show that there exists a 2-periodic orbit corresponding to a maximum of H_2.

3. Show that there is another, geometrically distinct 2-periodic orbit.
