Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 11 (due Thursday 21.01.16)

Problem 1 (4 points)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded and consider the problem

$$\begin{aligned} -\operatorname{div}(a\nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where $f \in H^{-1}(\Omega) = (H_0^1(\Omega))^*$ and $a \in L^\infty(\Omega; \mathbb{R}^{n\times n})$ such that

$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq c_0|\xi|^2 \qquad \text{for a. e. } x \in \Omega \text{ and all } \xi \in \mathbb{R}^n.$$

Use the continuation method to show that there exists a unique weak solution $u \in H_0^1(\Omega)$ to (1). You may use without proof that in the case $a_{ij} = c\delta_{ij}$ with $c > 0$ a unique weak solution exists.

Problem 2 (4 points)
Let $0 < \lambda < n+2$ and $\Omega \subset \mathbb{R}^n$ be open. Show that $u_k \rightharpoonup u$ in $L^2_{\text{loc}}(\Omega)$ implies

$$[u]_{\mathcal{L}^{2,\lambda}(\Omega)} \leq \liminf_{k\to\infty} [u_k]_{\mathcal{L}^{2,\lambda}(\Omega)},$$

where both sides may take the value $+\infty$.
Hint: Passing to a subsequence one may assume that $L := \lim_{k\to\infty}[u_k]_{\mathcal{L}^{2,\lambda}(\Omega)} < \infty$. If the claim is not true, then there are $\delta > 0$, $x_0 \in \Omega$ and $\rho > 0$ such that

$$\rho^{-\lambda}\int_{\Omega(x_0,\rho)} |u - u_{x_0,\rho}|^p > L^2 + \delta.$$

Derive a contradiction to the weak convergence of u_k.

Please turn over.

Problem 3 (8 points)

Assume that the coefficients $A_{ij}^{\alpha\beta}$ are constant and satisfy the Legendre-Hadamard condition $A_{ij}^{\alpha\beta}\zeta_\alpha\zeta_\beta\eta^i\eta^j \geq \nu|\zeta|^2|\eta|^2$ for all $\zeta \in \mathbb{R}^n$, $\eta \in \mathbb{R}^N$.

1. Let $f \in L^2(\mathbb{R}^n; \mathbb{R}^{n\times N})$. Show that there exists a weak solution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^N)$ with $\nabla u \in L^2(\mathbb{R}^n; \mathbb{R}^{n\times N})$ of

$$-D_\alpha(A_{ij}^{\alpha\beta} D_\beta u^j) = -D_\alpha f_\alpha^i, \quad i = 1, \ldots, N \qquad \big(-\mathrm{div} A\nabla u = -\mathrm{div} f\big),$$

which is uniquely determined up to constants and satisfies

$$\int_{\mathbb{R}^n} |\nabla u|^2 \,\mathrm{d}x \leq \frac{1}{\nu^2} \int_{\mathbb{R}^n} |f|^2 \,\mathrm{d}x. \tag{2}$$

Hints:

(a) *For existence, first obtain a unique solution* $u_k \in W^{1,2}(\mathbb{R}^n; \mathbb{R}^N)$ *of* $-\mathrm{div} A\nabla u_k + \frac{1}{k} u_k = -\mathrm{div} f$. *Then, deduce that a subsequence of* ∇u_k *converges weakly in* L^2 *and hence* $u_k - (u_k)_{0,1}$ *converges weakly in* $W^{1,2}_{\mathrm{loc}}$ *to the desired solution.*

(b) *For uniqueness, use that reverse Poincaré estimates*

$$\sup_{B_{r/2}} |\nabla u|^2 \leq \frac{C}{r^{n+2}} \int_{B_r} |u - u_r|^2 \,\mathrm{d}x \leq \frac{C}{r^n} \int_{B_r} |\nabla u|^2 \,\mathrm{d}x$$
$$\sup_{B_{r/2}} |\nabla^2 u|^2 \leq \frac{C}{r^{n+2}} \int_{B_r} |\nabla u - (\nabla u)_r|^2 \,\mathrm{d}x$$

are true for arbitrary balls.

2. Let $0 < \lambda < n + 2$ and $f \in L^2_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^{n\times N})$ with $[f]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} < \infty$. Show that there exists a weak solution $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^N)$ with $[\nabla u]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} < \infty$ of

$$-\mathrm{div} A\nabla u = -\mathrm{div} f,$$

which is uniquely determined up to an affine function and satisfies

$$[\nabla u]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} \leq C[f]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)}.$$

Hints:

(a) *Assume first that* $\mathrm{spt}\, f \subset B_{R_0}(0)$ *so that* $f \in L^2$. *Let* u *be the solution from 1. and deduce the estimate for* $[\nabla u]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)}$ *from interior estimates and (2).*

(b) *For general* f *let* $\eta \in C_c^\infty(B_1(0))$ *with* $\eta = 1$ *on* $B_{1/2}(0)$ *and set* $\eta_k(x) = \eta(2^{-k}x)$. *Let* $f_k = \eta_k(f - f_{0,2^k})$ *and let* u_k *be the solution with right hand side* f_k. *Show that* $[f_k]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} \leq [f]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)}$ *and* $[\nabla u_k]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)} \leq C[f]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)}$ *and deduce that for* $\rho \geq 1$ *we have*

$$\int_{B_\rho(0)} |\nabla u_k - (\nabla u_k)_{0,\rho}|^2 \,\mathrm{d}x \leq C\rho^\lambda [f]^2_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)},$$
$$|(\nabla u_k)_{0,\rho} - (\nabla u_k)_{0,1}| \leq C(\rho)[f]_{\mathcal{L}^{2,\lambda}(\mathbb{R}^n)}.$$

Conclude weak convergence of (a subsequence of) $\nabla u_k - (\nabla u_k)_{0,1}$ *and* $u_k - (u_k)_{0,1} - (\nabla u_k)_{0,1}x$.

Total: 16 points