Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 08 (due Thursday 17.12.15)

Problem 1 (Minimal graphs of revolution, 6 points)
Given $a, b > 0$ and $u \in C^1([-1,1])$ such that $u > 0$, $u(-1) = a$, $u(1) = b$, the area of the surface obtained by rotating the graph of u about the x-axis is

$$A(u) = 2\pi \int_{-1}^{+1} u\sqrt{1 + |u'|^2}\,\mathrm{d}x$$

1. Derive the Euler-Langrange equation of A.
2. If u is sufficiently regular and minimizes A in the class of functions with the same boundary values, show that
$$u^2 = c^2(1 + u'^2) \tag{1}$$
for some constant $c \in \mathbb{R}$.
3. Solve (1). Is the solution a minimizer of the area among surfaces of revolution with the same boundary values? Is there always a minimizer among smooth graphs with prescribed boundary values?

Problem 2 (Null Lagrangian, 3 points)
Show that

$$L(P) = \operatorname{trace}(P^2) - (\operatorname{trace} P)^2, \qquad P \in \mathbb{R}^{n\times n}$$

is a null Lagrangian.

Problem 3 (No Brouwer in infinite-dimensional spaces, 4 points)
Let H be an infinite-dimensional separable Hilbert space. Given a complete orthonormal system (e_k), define a map $T\colon H \to H$ by $e_k \mapsto e_{k+1}$, that is

$$x = \sum_{k=1}^{\infty} \alpha_k e_k \mapsto Tx = \sum_{k=1}^{\infty} \alpha_k e_{k+1}.$$

Show that

$$f(x) = \frac{1}{2}(1 - \|x\|)e_1 + Tx$$

maps the closed unit ball continuously into itself, but does not have a fixed point.

Problem 4 (Non-existence of minimizer, 3 points)
Let $M = \{u \in W^{1,p}(0,1) : u(0) = 0,\ u(1) = 1\}$, $p \in [1,\infty]$ and

$$F(u) = \int_0^1 \sqrt[4]{1 + u'(x)^2}\,\mathrm{d}x.$$

Show that $\inf_{u\in M} F(u) = 1$ and that F has no minimizer in M.

Total: 16 points