Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 07 (due Thursday 10.12.15)

Problem 1 (A priori estimates via maximum principle, 4 points)
Let $Q = (0,1)^n$ be the unit cube in $\mathbb{R}^n$.

1. Let $C_1 > 0, C_2 > 0$ be two constants and show that a weak solution $u \in H^1(Q)$ of
$$\begin{aligned} -\Delta u &= C_1 \quad \text{in } Q, \\ u &= C_2 \quad \text{on } \partial Q \end{aligned}$$
satisfies
$$\|u\|_{L^\infty(Q)} \leq C_1 + C_2.$$

2. Let $f \in L^\infty(Q)$ and $g \in L^\infty(\partial Q)$. Show that a weak solution $u \in H^1(Q)$ of
$$\begin{aligned} -\Delta u &= f \quad \text{in } Q, \\ u &= g \quad \text{on } \partial Q \end{aligned}$$
satisfies
$$\|u\|_{L^\infty(Q)} \leq C\left(\|f\|_{L^\infty(Q)} + \|g\|_{L^\infty(\partial Q)}\right)$$
with some constant $C > 0$.

Problem 2 (A gradient estimate, 4 points)
Let $\Omega \subset \mathbb{R}^n$ be open and smoothly bounded and assume that u is a smooth solution of $Lu = -\sum_{i,j} a^{ij} D_{ij} u = 0$ in Ω where the $a^{ij} \colon \Omega \to \mathbb{R}$ are smooth and L is uniformly elliptic, i. e. there is $\theta > 0$ such that
$$\sum_{i,j=1}^n a^{ij} \xi_i \xi_j \geq \theta |\xi| \qquad \text{for all} \qquad \xi \in \mathbb{R}^n.$$
Show that $v = |Du|^2 + \lambda |u|^2$ satisfies $Lv \leq 0$ for λ sufficiently large and deduce
$$\|Du\|_{L^\infty(\Omega)} \leq C\Big(\|Du\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\Big)$$
where C depends on the dimension and the a^{ij}.
Hint: Before applying a maximum principle from the lecture, note that L is not in divergence form here.

Problem 3 (Harnack inequality for harmonic functions, 4 points)
Let $\Omega \subset \mathbb{R}^n$ be open.

1. Without using any result from the lecture show that for each connected and open set $V \Subset \Omega$ there is a constant C such that the Harnack inequality
$$\sup_V u \leq C \inf_V u$$
is true for any harmonic function $u \colon \Omega \to \mathbb{R}$, $u \geq 0$.

   Hint: Mean value property.

2. Assume in addition that Ω is connected and let (u_k) be a nondecreasing sequence of harmonic functions on Ω such that there is $y \in \Omega$ with $\sup_{n \in \mathbb{N}} u_n(y) < \infty$. Show that u_n converges locally uniformly to a harmonic function u.

Problem 4 (Comparison principle for a nonlinear problem, 4 points)
Let $\Omega \subset \mathbb{R}^n$ be open and smoothly bounded and assume that $u, v \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfy

$$\mathcal{F}[u] \geq \mathcal{F}[v] \quad \text{in } \Omega, \qquad u \leq v \quad \text{on } \partial\Omega,$$

where $\mathcal{F}[u](x) = F(x, u(x), Du(x), D^2u(x))$ with a smooth function $F\colon \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n} \to \mathbb{R}$, $F\colon (x, z, p, r) \mapsto F(x, z, p, r)$ such that

- the matrix $\frac{\partial}{\partial r_{ij}} \mathcal{F}[u_\theta](x)$ is uniformly positive definite for $x \in \Omega$ and $u_\theta = \theta u + (1-\theta)v$, $\theta \in [0, 1]$;
- F is non-increasing in z for all x, p, r.

Show that $u \leq v$ in Ω.

Total: 16 points