Nonlinear Partial Differential Equations I — University of Bonn
Winter term 2015/2016 — Prof. Dr. J. J. L. Velázquez
Problem Sheet 05 (due Thursday 26.11.15)

Problem 1 (4 points)
Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p \leq \infty$.

1. Assume that $g\colon \mathbb{R} \to \mathbb{R}$ is continuously differentiable with $\|g'\|_\infty \leq C$ and satisfies $g(0) = 0$ if Ω has infinite measure. Show that $g \circ u \in W^{1,p}(\Omega)$ for all $u \in W^{1,p}(\Omega)$.

2. Conclude that

$$u^+ = \max(u, 0), \qquad u^- = -\min(u, 0) \qquad \text{and} \qquad |u|$$

belong to $W^{1,p}(\Omega)$ if $u \in W^{1,p}(\Omega)$ and find the corresponding weak derivatives.

Problem 2 (4 points)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

1. Suppose that $u \in H^1(\Omega)$ satisfies $u \geq 0$ on $\partial\Omega$ and

$$-\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \leq 0$$

for all $v \in H^1_0(\Omega)$ such that $v \geq 0$ a. e. in Ω. Show that $u \geq 0$ a. e. in Ω.

2. Let now u, v be smooth in $\overline{\Omega}$ and prove that
   (a) $\Delta u \leq 0$ in Ω and $u \geq \delta > 0$ on $\partial\Omega$ imply $u \geq \delta$ in Ω;
   (b) $-\Delta u \leq -\Delta v$ in Ω and $u \leq v$ on $\partial\Omega$ imply $u \leq v$ in Ω.

Problem 3 (8 points)
Let $\Omega \subset \mathbb{R}^n$ have finite measure and fix $p \in [1, \infty)$.

1. Let $u \in L^p(\Omega)$ and show that

$$\int_\Omega |u|^p \, \mathrm{d}x = p \int_0^\infty t^{p-1} |U_t| \, \mathrm{d}t$$

where $U_t = \{x \in \Omega : |u(x)| > t\}$.

2. Show that $L^p(\Omega) \subset$ w-$L^p(\Omega)$ with

$$|U_t| \leq \left(\frac{\|u\|_{L^p}}{t} \right)^p$$

for all $t > 0$ and all $u \in L^p(\Omega)$. Show that this inequality cannot be improved.

3. Show that w-$L^p(\Omega) \subset L^q(\Omega)$ for $p > 1$ and all $q \in [1, p)$.

4. Do we have $L^p(\Omega) \subsetneq$ w-$L^p(\Omega)$?

Total: 16 points