Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 03 (due Thursday 12.11.15)

Problem 1 (H^2-regularity for a semilinear problem, 3 points)
Let $u \in H^1(\mathbb{R}^n)$ have compact support and be a weak solution of

$$-\Delta u + c(u) = f \qquad \text{in } \mathbb{R}^n,$$

where $f \in L^2(\mathbb{R}^n)$ and $c \in C^1(\mathbb{R})$ with $c(0) = 0$, $c' \geq 0$. Show that $u \in H^2(\mathbb{R}^n)$.

Problem 2 (Discontinuous coefficients, 1+2+2 points)
Consider the problem

$$\begin{aligned} (au')' &= 2 \text{ in } (-1,1), \\ u(\pm 1) &= 0, \end{aligned}$$

where

$$a(x) = \begin{cases} a_- & : x < 0, \\ a_+ & : x > 0 \end{cases}$$

and $a_-, a_+ \in \mathbb{R}_{>0}$.

1. Show that there exists a unique weak solution $u \in H^1((-1,1))$.
2. Compute the weak solution.

   Hint: Compute a smooth solution in $(-1,0)$ and $(0,1)$ and investigate transmission conditions at $x = 0$.
3. Under what condition(s) is u' continuous?

Problem 3 (Boundary regularity, 4+4 points)
Let (r, φ) be polar coordinates in $\mathbb{R}^2$ and $\Omega_\omega = \{(r\cos\varphi, r\sin\varphi) : 0 < r < 1,\ 0 < \varphi < \omega\} \subset \mathbb{R}^2$, $0 < \omega < 2\pi$.

1. Verify that $u(r,\varphi) = r^\alpha \sin\alpha\varphi$ with $\alpha = \pi/\omega$ is a weak solution of

   $$\begin{aligned} \Delta u &= 0 && \text{in } \Omega_\omega, \\ u &= 0 && \text{on } \Gamma_D = \{(r\cos\varphi, r\sin\varphi) : 0 \leq r \leq 1, \varphi \in \{0, \omega\}\}, \\ \nabla u \cdot \nu &= \alpha \sin\alpha\varphi && \text{on } \Gamma_N = \partial\Omega \setminus \Gamma_D. \end{aligned}$$

   For which ω do we have $u \in H^2(\Omega_\omega)$? If $u \notin H^2(\Omega_\omega)$ what is the problem?
2. Verify that $u(r,\varphi) = r^{\alpha/2}\cos(\frac{\alpha}{2}\varphi)$ with $\alpha = \pi/\omega$ is a weak solution of

   $$\begin{aligned} \Delta u &= 0 && \text{in } \Omega_\omega, \\ u &= 0 && \text{on } \Gamma_D = \{(r\cos\omega, r\sin\omega) : 0 \leq r \leq 1\}, \\ \nabla u \cdot \nu &= 0 && \text{on } \Gamma_{N_1} = (0,1) \times \{0\}, \\ \nabla u \cdot \nu &= \frac{\alpha}{2}\cos\frac{\alpha}{2}\varphi && \text{on } \Gamma_{N_2} = \{(\cos\varphi, \sin\varphi) : 0 < \varphi < \omega\} \end{aligned}$$

   and determine whether $u \in H^2(\Omega_\omega)$.

Total: 16 points