Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 02 (due Thursday 05.11.15)

Problem 1 (A Robin boundary condition, 2+2 points)
Let $\Omega = B_1(0) \subset \mathbb{R}^2$, $f \in L^2(\Omega)$, and for $\beta \in \mathbb{R}$ consider the problem

$$\begin{aligned} -\Delta u &= f \quad && \text{in } \Omega, \\ \nabla u \cdot \nu &= \beta u \quad && \text{on } \partial\Omega. \end{aligned}$$

1. Derive a weak formulation of the problem and discuss existence and uniqueness of weak solutions for $\beta \leq 0$.
2. Let $f = 0$. Show that there is an infinite number of positive β's such that the problem has a nontrivial solution.

   Hint: Polar coordinates.

Problem 2 (Rank-one convexity, 2 points)
Let $F : \mathbb{R}^{N\times n} \to \mathbb{R}$ be twice continuously differentiable. Show that F is rank-one convex, that is, the mapping $t \mapsto F(p + tq)$ is convex for all $p, q \in \mathbb{R}^{N\times n}$ with $\operatorname{rank}(q) = 1$, if and only if D^2F satisfies the Legendre-Hadamard-like condition

$$\sum_{\alpha,\beta=1}^{n} \sum_{i,j=1}^{N} \partial_{p_{i\alpha}} \partial_{p_{j\beta}} F(p) \xi_\alpha \xi_\beta \eta_i \eta_j \geq 0$$

for all $p \in \mathbb{R}^{N\times n}$, all $\xi \in \mathbb{R}^n$ and all $\eta \in \mathbb{R}^N$.

Problem 3 (3+1 points)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f \in L^2(\Omega)$ and $b \in L^\infty(\Omega; \mathbb{R}^n)$. Define

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v + b \cdot \nabla u \, v \, \mathrm{d}x \qquad u, v \in H_0^1(\Omega).$$

1. Suppose that for $u \in H_0^1(\Omega)$ with $B(u, v) = 0$ for all $v \in H_0^1(\Omega)$ we have $u = 0$. Show that there is a unique $w \in H^1(\Omega)$ such that

   $$B(w, v) = \int_\Omega f v \, \mathrm{d}x$$

   for all $v \in H_0^1(\Omega)$.

   Hint: Fredholm alternative.
2. Suppose that b satisfies $\operatorname{div} b \leq 0$ in the sense $\int_\Omega \nabla v \cdot b \, \mathrm{d}x \geq 0$ for all $v \in C_c^\infty(\Omega)$, $v \geq 0$ almost everywhere. Show that there is a unique weak solution to the boundary value problem

   $$\begin{aligned} -\Delta u + b \cdot \nabla u &= f \quad && \text{in } \Omega, \\ u &= 0 \quad && \text{on } \partial\Omega. \end{aligned}$$

Please turn over.

Problem 4 (Difference quotients, 2+2+2 points)
Let $\Omega \subset \mathbb{R}^n$ be a connected, open and bounded domain with smooth boundary. For $u\colon \Omega \to \mathbb{R}$, $i \in \{1, \dots, n\}$, $h \in \mathbb{R}$, $V \Subset U$ such that $0 < |h| < \operatorname{dist}(V, \partial\Omega)$, and $x \in V$ denote by

$$D_i^h u(x) = \frac{u(x + he_i) - u(x)}{h}$$

the i-th difference quotient quotient.

1. Let $1 \leq p < \infty$ and $u \in W^{1,p}(\Omega)$. Show that for each $V \Subset \Omega$
$$\|D_i^h u\|_{L^p(V)} \leq \|D_i u\|_{L^p(\Omega)}$$
for all $0 < |h| < \operatorname{dist}(V, \partial\Omega)$.
2. Let $1 < p < \infty$, $u \in L^p(V)$, $V \Subset \Omega$, and assume there is a constant $C > 0$ such that
$$\|D_i^h u\|_{L^p(V)} \leq C$$
for all $0 < |h| < \operatorname{dist}(V, \partial\Omega)$. Show that the weak derivative $D_i u$ exists in V and satisfies $\|D_i u\|_{L^p(V)} \leq C$.
3. Show that 2. is false for $p = 1$.

Total: 16 points