Nonlinear Partial Differential Equations I University of Bonn
Winter term 2015/2016 Prof. Dr. J. J. L. Velázquez
Problem Sheet 01 (due Thursday 29.10.15)

Problem 1 (Banach's Fixed Point Theorem, 2 points)
Let (X, d) be a complete metric space, $M \subset X$ non-empty and closed, and $T\colon M \to M$ a contractive mapping, that is, there is a constant $\kappa \in (0, 1)$ such that

$$d(Tx, Ty) \leq \kappa d(x, y)$$

for all $x, y \in M$. Show that T has a unique fixed point $x \in M$ and that any sequence $x_k = Tx_{k-1}$, $x_0 \in M$ converges to x as $k \to \infty$.

Problem 2 (Lax-Milgram, 4 points)
Let H be a real Hilbert space, $F \in H^*$ a bounded linear functional on H, and $B\colon H \times H \to \mathbb{R}$ a bilinear form which is bounded and coercive, that is, there are constants $M \geq \alpha > 0$ such that

$$B(u, v) \leq M\|u\|\|v\| \qquad \text{and} \qquad B(u, u) \geq \alpha\|u\|^2 \qquad \text{for all } u, v \in H.$$

Use Banach's Fixed Point Theorem to prove that there is a unique $u \in H$ such that

$$B(u, v) = \langle F, v\rangle \qquad \text{for all } v \in H.$$

Hint: Use the Riesz Representation Theorem to define an appropriate map $T\colon H \to H$.

Problem 3 (Poisson Equation, 2+4 points)
Let Ω be open and bounded with smooth boundary.

1. Let $f \in L^2(\Omega)$ and show that the problem

$$\begin{aligned} -\Delta w &= f \quad \text{in } \Omega, \\ w &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

   has a unique weak solution $u \in H_0^1(\Omega)$, and that this solution satisfies

$$\|\nabla u\|_{L^2} \leq C\|f\|_{L^2}.$$

   Note: This is basic linear PDE theory. If necessary, you should familiarize yourself with it quickly. See for instance the book of Evans.

2. Let $f\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz continuous. Show that

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

   has a unique weak solution $u \in H_0^1(\Omega)$ provided that the Lischitz constant of f is sufficiently small.

   Hint: Define an appropriate map $T\colon L^2(\Omega) \to L^2(\Omega)$ and use Banach's Fixed Point Theorem.

Please turn over.

Problem 4 (1+3 points)
Let $\Omega \subset \mathbb{R}^n$ be open and bounded with smooth boundary.

1. Show that the unique solution of
$$\begin{aligned} \Delta u &= u^3 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$
in $C^2(\Omega) \cap C^0(\overline{\Omega})$ is $u \equiv 0$.

2. Show that the problem
$$\begin{aligned} \Delta u &= u^2 && \text{in } \Omega, \\ u &\geq 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega \end{aligned}$$
has a unique solution in $C^2(\Omega) \cap C^0(\overline{\Omega})$. Is the same true if the constraint $u \geq 0$ is dropped?

Total: 16 points