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PDE and Modelling

Exercise sheet 10 – Optional

This exercise sheet is optional, and does not count towards the score for the exercise sessions.

Problem 1

Let $\Omega \subset \mathbb{R}^n$ be open and bounded. For $u \in W_0^{2,2}(\Omega)$, consider

$$I[u] = \int_\Omega (\Delta u)^2 - 2fu\,\mathrm{d}x$$

(a) Compute the first variation of I, and show that the Euler-Lagrange equation is given by

$$\Delta^2 u = f \qquad \text{in } \Omega.$$

(b) Compute the second variation of I.

Problem 2

Let $f : \mathbb{R}^{n\times m} \to \mathbb{R} \cup \{+\infty\}$ be a function.

(a) Show that f is rank-1 convex if and only if the restriction $\phi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\phi(t) = f(A + tu \otimes v)$$

for any $A \in \mathbb{R}^{n\times m}$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^m$.

(b) Show that the function $\det : \mathbb{R}^{2\times 2} \to \mathbb{R}$ is rank-1 convex, but not convex.

Problem 3

Let $\Omega \subset \mathbb{R}^n$ be a smooth, bounded open domain, and let $1 < p < \infty$. For $f \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, define

$$I[u] = \int_\Omega \frac{1}{p}|\nabla u|^p - fu\,\mathrm{d}x.$$

(a) Suppose $\{u_k\}_{k\in\mathbb{N}}$ is a sequence in $W_0^{1,p}(\Omega)$ such that $I[u_k]$ is bounded. Show that u_k is bounded in $W_0^{1,p}(\Omega)$.

(b) Suppose that u_k converges to u weakly in $W_0^{1,p}(\Omega)$, that is, ∇u_k and u_k converge weakly in L^p to ∇u and u, respectively. Show that

$$I[u] \le \liminf_{k\to\infty} I[u_k].$$

(c) Derive from the previous parts that I has a unique minimizer in $W_0^{1,p}(\Omega)$.

(d) Show that the equation

$$-\Delta_p u := -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f$$

has a solution $u \in W_0^{1,p}(\Omega)$ in the distributional sense.