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PDE and Modelling

Exercise sheet 6

Problem 1

(a) Suppose

$$v(x,t) = (\gamma x_1, -\gamma x_2, v_3(x_1,t))$$

Derive an equation for v_3 such that v satisfies the Navier-Stokes equation with $p(x,t) = -\frac{\gamma^2}{2}(x_1^2 + x_2^2)$.

(b) Recall that the vorticity $\omega = \operatorname{curl} v$ satisfies the evolution equation $D_t\omega = S\omega + \nu\Delta\omega$, where S is the symmetric part of Dv. Show that the vorticity ω is of the form $\omega = (0, \omega_2(x_1,t), 0)$ if the velocity v is of the form given above, and derive an equation for ω_2.

(c) Suppose ω_2 is given. Can one determine v_3? If yes, how?

Problem 2

Consider the stationary flow, that is $\frac{\partial v}{\partial t} = 0$ of a viscous fluid between two stationary rigid boundaries $\{x_2 = \pm h\}$ under a constant pressure gradient $\nabla p = (-P, 0, 0)$. Derive a solution $v(x) = u(x_2)$ of the stationary incompressible Navier-Stokes equation

$$\begin{cases} (v\cdot\nabla)v - \nu\Delta v = -\dfrac{1}{\rho}\nabla p \\ \operatorname{div} v = 0 \end{cases}$$

with boundary conditions $v(x) = 0$ if $x_2 = \pm h$ (remember that ρ is constant due to incompressibility).

Hint: Use incompressibility the boundary conditions to derive $u_2 = 0$. Then, by the Navier Stokes equation deduce that $u_3 = 0$.

Problem 3

Let $A \in \mathbb{R}^{3\times 3}$, and suppose that $\operatorname{tr} A = 0$. The Levi-Civita symbol ε is given by $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$ and $\varepsilon_{ijk} = 0$ in all other cases. Finally, denote

$$S = \frac{A + A^T}{2}, \qquad W = \frac{A - A^T}{2}, \qquad \omega_l = \sum_{j,k=1}^{3} \varepsilon_{ljk} W_{kj}.$$

(a) Show that $A^2 = SW - (SW)^T + S^2 + W^2$, and use this identity to show that

$$\sum_{j,k,l=1}^{3} \varepsilon_{ijk} A_{lj} A_{kl} = 2\left((SW)_{i-1,i+1} - (SW)_{i+1,i-1}\right)$$

where indices are understood modulo 3.

(b) Show that $\omega_l = 2W_{l-1,l+1}$, and combine this with $\operatorname{tr} A = 0$ to show

$$\sum_{j,k,l=1}^{3} \varepsilon_{ijk} A_{lj} A_{kl} = -(S\omega)_i \tag{*}$$

Problem 4

Let $\Omega \subset \mathbb{R}^2$ be a domain, and suppose that $v : \Omega \to \mathbb{R}^2$ is a smooth vector field. Define $\omega : \Omega \to \mathbb{R}$ by $\omega = \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}$. Show that

$$D_t \omega = \nu \Delta \omega$$

if v solves the incompressible Navier-Stokes equations.

Hint: Define a suitable vector field in $\Omega \times \mathbb{R}$.

Due: Wednesday, June 12 at the end of the lecture