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PDE and Modelling

Exercise sheet 1

Problem 1

For a matrix $A \in \mathbb{R}^{n\times n}$ show the identity:

$$\frac{\partial \det A}{\partial A_{ij}} = (\operatorname{cof} A)_{ij}.$$

Hereby, cof A is the cofactor matrix given by $(\operatorname{cof} A)_{ij} := (-1)^{i+j} \det A^{ij}$, where $(n-1)\times(n-1)$ matrix $A^{ij} \in$ is the minor obtained from A by deleting the i-th row and j-th column.

Problem 2

Let $x : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ be the map from Lagrangian to Eulerian coordinates of the form

$$x(t, X) = (X_1 + tX_2, X_2 - t, e^{-t}X_3).$$

(a) Calculate the corresponding velocity field $v(t, x)$ in Eulerian coordinates.

(b) Verify for the function $\varphi(t, x) = x_1 + x_2 \cos t$ the formula

$$\frac{D}{Dt}\varphi(t, x) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi(t, x(t, X)),$$

where $\frac{D}{Dt}$ is the material derivative.

(c) Let $v : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the velocity field given by

$$v(t, x) = (tx_2, tx_1).$$

Calculate the streamlines and the trajectories.

Problem 3

Let $\Omega \subset \mathbb{R}^n$ open and convex, and let $u : \Omega \to \mathbb{R}^n$ be a C^2 vector field. A matrix $A \in \mathbb{R}^{n\times n}$ is called *skew-symmetric* or *skew* if $A^T = -A$. Prove that the following are equivalent:

(i) u is an infinitesimal rigid displacement, i.e., ∇u is constant and skew,

(ii) For all $p, q \in \Omega$,

$$(q - p) \cdot (u(q) - u(p)) = 0,$$

(iii) $\nabla u(p)$ is skew at each $p \in \Omega$.

Hint for (iii)⇒(i): Consider the derivative of $\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$ with respect to x_k, consider cyclic permutations of i, j and k, and make use of an appropriate combination of the equalities.

Problem 4

Let $x : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a smooth map from Lagrangian to Eulerian coordinates, and let $v(t, x)$ be the corresponding velocity field in Eulerian coordinates. Recall that

$$\mathrm{SO}(n) = \left\{ F \in \mathbb{R}^{n \times n} : F^T F = \mathrm{Id}, \det F = 1 \right\}.$$

(a) Assume that $\nabla x(t, X) \in \mathrm{SO}(n)$ for all $t \geq 0$, $X \in \mathbb{R}^n$. Show that $\nabla v(t, x)$ is skew.

(b) Show that $x(t, .)$ is a rigid body movement, that is, $x(t, X) = R(t)X + c(t)$ where $R(t) \in \mathrm{SO}(n)$.

Hint: Use the results of Problem 3.

Due: Friday, April 17 at the end of the lecture
